7

CHAPTER

Applications in Heat Transfer

7.1 INTRODUCTION

In this chapter, the Galerkin method introduced in Chapter 5 and the interpolation function concepts of Chapter 6 are applied to several heat transfer situations. Conduction with convection is discussed for one-, two-, and three-dimensional problems. Boundary conditions and forcing functions include prescribed heat flux, insulated surfaces, prescribed temperatures, and convection. The one-dimensional case of heat transfer with mass transport is also developed. The three-dimensional case of axial symmetry is developed in detail using appropriately modified two-dimensional elements and interpolation functions. Heat transfer by radiation is not discussed, owing to the nonlinear nature of radiation effects. However, we examine transient heat transfer and include an introduction to finite difference techniques for solution of transient problems.

7.2 ONE-DIMENSIONAL CONDUCTION: QUADRATIC ELEMENT

Chapter 5 introduced the concept of one-dimensional heat conduction via the Galerkin finite element method. In the examples of Chapter 5, linear, two-node finite elements are used to illustrate the concepts involved. Given the development of the general interpolation concepts in Chapter 6, we now apply a higher-order (quadratic) element to a previous example to demonstrate that (1) the basic procedure of element formulation is unchanged, (2) the system assembly procedure is unchanged, and (3) the results are quite similar.

EXAMPLE 7.1

Solve Example 5.4 using two, three-node line elements with equally spaced nodes. The problem and numerical data are repeated here as Figure 7.1a.

■ Solution

Per Equation 5.62, the element equations are

$$k_x A N_i(x) \left.\frac{\mathrm{d}T}{\mathrm{d}x}\right|_{x_1}^{x_2} - k_x A \int_{x_1}^{x_2} \frac{\mathrm{d}N_i}{\mathrm{d}x}\frac{\mathrm{d}T}{\mathrm{d}x}\,\mathrm{d}x + A\int_{x_1}^{x_2} Q N_i(x)\,\mathrm{d}x = 0 \qquad i = 1, 3$$

where now there are three interpolation functions per element.

The interpolation functions for a three-node line element are, per Equations 6.23–6.25

$$N_1(s) = 2\left(s - \frac{1}{2}\right)(s-1)$$

$$N_2(s) = -4s(s-1)$$

$$N_3(s) = 2s\left(s - \frac{1}{2}\right)$$

The components of the conductance matrix are then calculated as

$$k_{ij} = k_x A \int_{x_1}^{x_2} \frac{\mathrm{d}N_i}{\mathrm{d}x}\frac{\mathrm{d}N_j}{\mathrm{d}x}\,\mathrm{d}x \qquad i, j = 1, 3$$

and the heat generation vector components are

$$f_{Qi} = A\int_{x_1}^{x_2} Q N_i\,\mathrm{d}x \qquad i = 1, 3$$

and all f_Q components are zero in this example, as there is no internal heat source.

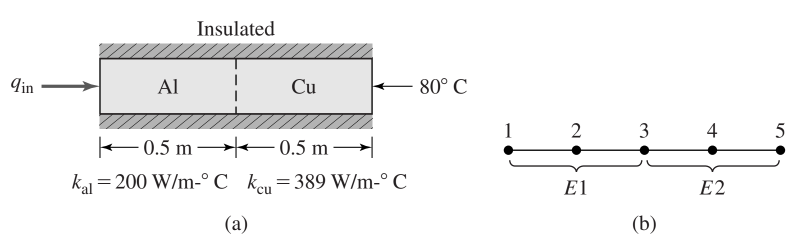

Figure 7.1
(a) Geometry and data for Example 7.1, outside diameter = 60 mm, $q_{in} = 4000$ W/m^2. (b) Two-element model, using quadratic elements.

In terms of the dimensionless coordinate $s = x/L$, we have $\mathrm{d}x = L\mathrm{d}s$ and $\mathrm{d}/\mathrm{d}x = (1/L)\,\mathrm{d}/\mathrm{d}s$, so the terms of the conductance matrix are expressed as

$$k_{ij} = \frac{k_x A}{L}\int_0^1 \frac{\mathrm{d}N_i}{\mathrm{d}s}\frac{\mathrm{d}N_j}{\mathrm{d}s}\,\mathrm{d}s \qquad i, j = 1, 3$$

The derivatives of the interpolation functions are

$$\frac{\mathrm{d}N_1}{\mathrm{d}s} = 4s - 3$$

$$\frac{\mathrm{d}N_2}{\mathrm{d}s} = 4(1 - 2s)$$

$$\frac{\mathrm{d}N_3}{\mathrm{d}s} = 4s - 1$$

Therefore, on substitution for the derivatives,

$$k_{11} = \frac{k_x A}{L}\int_0^1 (4s-3)^2\,\mathrm{d}s = \frac{k_x A}{L}\int_0^1 (16s^2 - 24s + 9)\,\mathrm{d}s$$

$$= \frac{k_x A}{L}\left(\frac{16s^3}{3} - 12s^2 + 9s\right)\Bigg|_0^1 = \frac{7k_x A}{3L}$$

Via mathematically identical procedures, the remaining terms of the conductance matrix are found to be

$$k_{12} = k_{21} = -\frac{8k_x A}{3L}$$

$$k_{13} = k_{31} = \frac{k_x A}{3L}$$

$$k_{22} = \frac{16k_x A}{3L}$$

$$k_{23} = k_{32} = -\frac{8k_x A}{3L}$$

$$k_{33} = \frac{7k_x A}{3L}$$

A two-element model with node numbers is shown in Figure 7.1b. Substituting numerical values, we obtain, for the aluminum half of the rod (element 1),

$$\left[k^{(1)}\right] = \frac{200(\pi/4)(0.006)^2}{3(0.5)}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} = \begin{bmatrix} 2.6389 & -3.0159 & 0.3770 \\ -3.0159 & 6.0319 & -3.0159 \\ 0.3770 & -3.0159 & 2.6389 \end{bmatrix}$$

and for the copper portion (element 2),

$$[k^{(2)}] = \frac{389(\pi/4)(0.006)^2}{3(0.5)} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} = \begin{bmatrix} 5.1327 & -5.8660 & 0.7332 \\ -5.8660 & 11.7320 & -5.8660 \\ 0.7332 & -5.8660 & 5.1327 \end{bmatrix}$$

At the internal nodes of each element, the flux terms are zero, owing to the nature of the interpolation functions $[N_2(x_1) = N_2(x_2) = 0]$. Similarly, at the junction between the two elements, the flux must be continuous and the equivalent "forcing" functions are zero. As no internal heat is generated, $Q = 0$, that portion of the force vector is zero for each element. Following the direct assembly procedure, the system conductance matrix is found to be

$$[K] = \begin{bmatrix} 2.6389 & -3.0159 & 0.3770 & 0 & 0 \\ -3.0159 & 6.0319 & -3.0159 & 0 & 0 \\ 0.3770 & -3.0159 & 7.7716 & -5.8660 & 0.7332 \\ 0 & 0 & -5.8660 & 11.7320 & -5.8660 \\ 0 & 0 & 0.7332 & -5.8660 & 5.1327 \end{bmatrix} \text{W/}^\circ\text{C}$$

and we note in particular that "overlap" exists only at the juncture between elements. The gradient term at node 1 is computed as

$$f_{g1} = -k_x A \left.\frac{\mathrm{d}T}{\mathrm{d}x}\right|_{x_1} = q_1 A = 4000\left(\frac{\pi}{4}\right)(0.06)^2 = 11.3097 \text{ W}$$

while the heat flux at node 5 is an unknown to be calculated via the system equations.

The system equations are given by

$$\begin{bmatrix} 2.6389 & -3.0159 & 0.3770 & 0 & 0 \\ -3.0159 & 6.0319 & -3.0159 & 0 & 0 \\ 0.3770 & -3.0159 & 7.7716 & -5.8660 & 0.7332 \\ 0 & 0 & -5.8660 & 11.7320 & -5.8660 \\ 0 & 0 & 0.7332 & -5.8660 & 5.1327 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ 80 \end{Bmatrix} = \begin{Bmatrix} 11.3097 \\ 0 \\ 0 \\ 0 \\ -Aq_5 \end{Bmatrix}$$

Prior to solving for the unknown nodal temperatures T_1–T_4, the nonhomogeneous boundary condition $T_5 = 80^\circ$C must be accounted for properly. In this case, we reduce the system of equations to 4×4 by transposing the last term of the third and fourth equations to the right-hand side to obtain

$$\begin{bmatrix} 2.3689 & -3.0159 & 0.3770 & 0 \\ -3.0159 & 6.0319 & -3.0159 & 0 \\ 0.3770 & -3.0159 & 7.7716 & -5.8660 \\ 0 & 0 & -5.8660 & 11.7320 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 11.3907 \\ 0 \\ -0.7332(80) \\ 5.8660(80) \end{Bmatrix}$$

$$= \begin{Bmatrix} 11.3097 \\ 0 \\ -58.6560 \\ 489.2800 \end{Bmatrix}$$

Solving the equations by Gaussian elimination (Appendix C), the nodal temperatures are

$$T_1 = 95.11^\circ\text{C}$$
$$T_2 = 90.14^\circ\text{C}$$
$$T_3 = 85.14^\circ\text{C}$$
$$T_4 = 82.57^\circ\text{C}$$

and the heat flux at node 5 is calculated using the fifth equation

$$-Aq_5 = 0.7332T_3 - 5.8660T_4 + 5.1327(80)$$

to obtain

$$q_5 = 4001.9 \text{ W/m}^2$$

which is observed to be in quite reasonable numerical agreement with the heat input at node 1.

The solution for the nodal temperatures in this example is identical for both the linear and quadratic interpolation functions. In fact, the solution we obtained is the exact solution (Problem 7.1) represented by a linear temperature distribution in each half of the bar. It can be shown [1] that, if an exact solution exists and the interpolation functions used in the finite element formulation include the terms appearing in the exact solution, then the finite element solution corresponds to the exact solution. In this example, the quadratic interpolation functions include the linear terms in addition to the quadratic terms and thus capture the exact, linear solution. The following example illustrates this feature in terms of the field variable representation.

EXAMPLE 7.2

For the quadratic field variable representation

$$\phi(x) = a_0 + a_1 x + a_2 x^2$$

determine the explicit form of the coefficients a_0, a_1, a_2 in terms of the nodal variables if the three nodes are equally spaced. Then use the results of Example 7.1 to show $a_2 = 0$ for that example.

■ Solution

Using the interpolation functions from Example 7.1, we can write the field variable representation in terms of the dimensionless variable s as

$$\phi(s) = (2s^2 - 3s + 1)\phi_1 + 4(s - s^2)\phi_2 + (2s^2 - s)\phi_3$$

Collecting coefficients of similar powers of s,

$$\phi(s) = \phi_1 + (4\phi_2 - 3\phi_1 - \phi_3)s + (2\phi_1 - 4\phi_2 + 2\phi_3)s^2$$

Therefore,

$$a_0 = \phi_1$$
$$a_1 = 4\phi_2 - 3\phi_1 - \phi_3$$
$$a_2 = 2\phi_1 - 4\phi_2 + 2\phi_3$$

Using the temperature results of Example 7.1 for the aluminum element, we have

$$\phi_1 = T_1 = 95.14$$
$$\phi_2 = T_2 = 90.14$$
$$\phi_3 = T_3 = 85.14$$
$$a_2 = 2(95.14) - 4(90.14) + 2(85.14) = 0$$

For element 2, representing the copper portion of the bar, the same result is obtained.

7.3 ONE-DIMENSIONAL CONDUCTION WITH CONVECTION

One-dimensional heat conduction, in which no heat flows from the surface of the body under consideration (as in Figure 5.8), is not commonly encountered. A more practical situation exists when the body is surrounded by a fluid medium and heat flow occurs from the surface to the fluid via *convection*. Figure 7.2a shows a solid body, which we use to develop a one-dimensional model of heat transfer including both conduction and convection. Note that the representation is the same as in Figure 5.8 with the very important exception that the assumption of an insulated surface is removed. Instead, the body is assumed to be surrounded by a fluid medium to which heat is transferred by convection. If the fluid is in motion as a result of some external influence (a fan or pump, for example), the convective heat transfer is referred to as *forced convection*. On the other hand, if motion of the fluid exists only as a result of the heat transfer taking place, we have *natural convection*. Figure 7.2b depicts a control volume of differential length, which is assumed to have a constant cross-sectional area and uniform

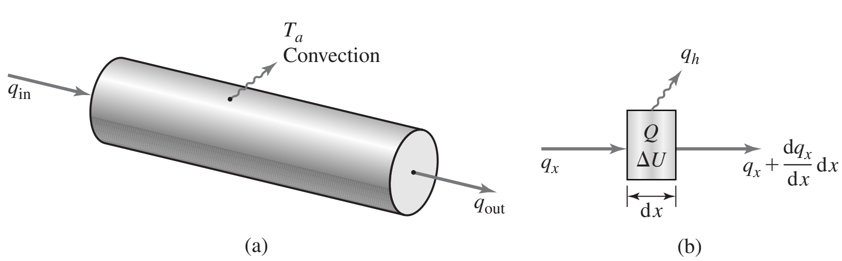

Figure 7.2 One-dimensional conduction with surface convection. (a) General model. (b) Differential element as a control volume.

material properties. The convective heat transfer across the surface, denoted q_h, represents the heat flow rate (heat flux) across the surface per unit *surface* area. To apply the principle of conservation of energy to the control volume, we need only add the convection term to Equation 5.54 to obtain

$$q_x A\,\mathrm{d}t + QA\,\mathrm{d}x\,\mathrm{d}t = \Delta U + \left(q_x + \frac{\partial q_x}{\partial x}\,\mathrm{d}x\right)A\,\mathrm{d}t + q_h P\,\mathrm{d}x\,\mathrm{d}t \tag{7.1}$$

where all terms are as previously defined except that P is the peripheral dimension of the differential element and q_h is the heat flux due to convection. The convective heat flux is given by [2]

$$q_h = h(T - T_a) \tag{7.2}$$

where

h = convection coefficient, W/(m^2-°C), Btu/(hr-ft^2-°F)

T = temperature of surface of the body

T_a = ambient fluid temperature

Substituting for q_h and assuming steady-state conditions such that $\Delta U = 0$, Equation 7.1 becomes

$$QA = A\frac{\mathrm{d}q_x}{\mathrm{d}x} + hP(T - T_a) \tag{7.3}$$

which, via Fourier's law Equation 5.55, becomes

$$k_x\frac{\mathrm{d}^2T}{\mathrm{d}x^2} + Q = \frac{hP}{A}(T - T_a) \tag{7.4}$$

where we have assumed k_x to be constant.

While Equation 7.4 represents the one-dimensional formulation of conduction with convection, note that the temperature at any position x along the length of the body is not truly constant, owing to convection. Nevertheless, if the cross-sectional area is small relative to the length, the one-dimensional model can give useful results if we recognize that the computed temperatures represent average values over a cross section.

7.3.1 Finite Element Formulation

To develop the finite element equations, a two-node linear element for which

$$T(x) = N_1(x)T_1 + N_2(x)T_2 \tag{7.5}$$

is used in conjunction with Galerkin's method. For Equation 7.4, the residual equations (in analogy with Equation 5.61) are expressed as

$$\int_{x_1}^{x_2} \left[k_x\frac{\mathrm{d}^2T}{\mathrm{d}x^2} + Q - \frac{hP}{A}(T - T_a)\right] N_i(x)A\,\mathrm{d}x = 0 \qquad i = 1, 2 \tag{7.6}$$

or

$$A\int_{x_1}^{x_2} k_x \frac{\mathrm{d}^2T}{\mathrm{d}x^2} N_i(x)\,\mathrm{d}x - hP\int_{x_1}^{x_2} T(x)N_i(x)\,\mathrm{d}x + A\int_{x_1}^{x_2} QN_i(x)\,\mathrm{d}x + hPT_a\int_{x_1}^{x_2} N_i(x)\,\mathrm{d}x = 0 \qquad i = 1, 2 \tag{7.7}$$

Integrating the first term by parts and rearranging,

$$k_xA\int_{x_1}^{x_2} \frac{\mathrm{d}N_i}{\mathrm{d}x}\frac{\mathrm{d}T}{\mathrm{d}x}\,\mathrm{d}x + hP\int_{x_1}^{x_2} T(x)N_i(x)\,\mathrm{d}x = A\int_{x_1}^{x_2} QN_i(x)\,\mathrm{d}x + hPT_a\int_{x_1}^{x_2} N_i(x)\,\mathrm{d}x + k_xAN_i(x)\left.\frac{\mathrm{d}T}{\mathrm{d}x}\right|_{x_1}^{x_2} \qquad i = 1, 2 \tag{7.8}$$

Substituting for $T(x)$ from Equation 7.5 yields

$$k_xA\int_{x_1}^{x_2} \frac{\mathrm{d}N_i}{\mathrm{d}x}\left(\frac{\mathrm{d}N_1}{\mathrm{d}x}T_1 + \frac{\mathrm{d}N_2}{\mathrm{d}x}T_2\right)\mathrm{d}x + hP\int_{x_1}^{x_2} N_i(x)[N_1(x)T_1 + N_2(x)T_2]\,\mathrm{d}x = A\int_{x_1}^{x_2} QN_i(x)\,\mathrm{d}x + hPT_a\int_{x_1}^{x_2} N_i(x)\,\mathrm{d}x + k_xAN_i(x)\left.\frac{\mathrm{d}T}{\mathrm{d}x}\right|_{x_1}^{x_2} \qquad i = 1, 2 \tag{7.9}$$

The two equations represented by Equation 7.9 are conveniently combined into a matrix form by rewriting Equation 7.5 as

$$T(x) = [N_1 \quad N_2]\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = [N]\{T\} \tag{7.10}$$

and substituting to obtain

$$k_xA\int_{x_1}^{x_2} \left[\frac{\mathrm{d}N}{\mathrm{d}x}\right]^T \left[\frac{\mathrm{d}N}{\mathrm{d}x}\right]\{T\}\,\mathrm{d}x + hP\int_{x_1}^{x_2} [N]^T[N]\{T\}\,\mathrm{d}x = A\int_{x_1}^{x_2} Q[N]^T\,\mathrm{d}x + hPT_a\int_{x_1}^{x_2} [N]^T\,\mathrm{d}x + k_xA\,[N]^T\left.\frac{\mathrm{d}T}{\mathrm{d}x}\right|_{x_1}^{x_2} \tag{7.11}$$

Equation 7.11 is in the desired finite element form:

$$\left[k^{(e)}\right]\{T\} = \left\{f_Q^{(e)}\right\} + \left\{f_h^{(e)}\right\} + \left\{f_g^{(e)}\right\} \tag{7.12}$$

where $[k^{(e)}]$ is the conductance matrix defined as

$$\left[k^{(e)}\right] = k_x A \int_{x_1}^{x_2} \left[\frac{\mathrm{d}N}{\mathrm{d}x}\right]^T \left[\frac{\mathrm{d}N}{\mathrm{d}x}\right] \mathrm{d}x + hP \int_{x_1}^{x_2} [N]^T [N]\,\mathrm{d}x \tag{7.13}$$

The first integral is identical to that in Equation 5.66, representing the axial conduction effect, while the second integral accounts for convection.

Without loss of generality, we let $x_1 = 0$, $x_2 = L$ so that the interpolation functions are

$$\begin{aligned} N_1 &= 1 - \frac{x}{L} \\ N_2 &= \frac{x}{L} \end{aligned} \tag{7.14}$$

The results of the first integral are as given in Equation 5.68, so we need perform only the integrations indicated in the second term (Problem 7.2) to obtain

$$\left[k^{(e)}\right] = \frac{k_x A}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{hPL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \left[k_c^{(e)}\right] + \left[k_h^{(e)}\right] \tag{7.15}$$

where $[k_c^{(e)}]$ and $[k_h^{(e)}]$ represent the conductive and convective portions of the matrix, respectively. Note particularly that both portions are symmetric.

The forcing function vectors on the right-hand side of Equation 7.12 include the internal heat generation and boundary flux terms, as in Chapter 5. These are given by

$$\left\{f_Q^{(e)}\right\} = A \begin{Bmatrix} \int_0^L QN_1\,\mathrm{d}x \\ \int_0^L QN_2\,\mathrm{d}x \end{Bmatrix} \tag{7.16}$$

$$\left\{f_g^{(e)}\right\} = k_x A \begin{Bmatrix} -\left.\dfrac{\mathrm{d}T}{\mathrm{d}x}\right|_0 \\ \left.\dfrac{\mathrm{d}T}{\mathrm{d}x}\right|_L \end{Bmatrix} = A\begin{Bmatrix} q_{x=0} \\ -q_{x=L} \end{Bmatrix} = A\begin{Bmatrix} q_1 \\ -q_2 \end{Bmatrix} \tag{7.17}$$

where q_1 and q_2 are the boundary flux values at nodes 1 and 2, respectively. In addition, the forcing function arising from convection is

$$\left\{f_h^{(e)}\right\} = hPT_a \begin{Bmatrix} \int_0^1 N_1\,\mathrm{d}x \\ \int_0^1 N_2\,\mathrm{d}x \end{Bmatrix} = \frac{hPT_aL}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{7.18}$$

where it is evident that the total element convection force is simply allocated equally to each node, like constant internal heat generation Q.

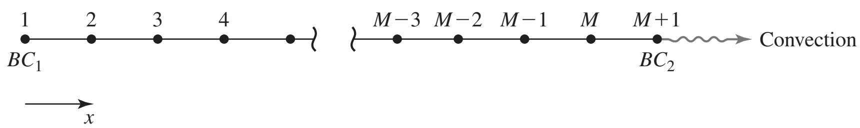

Figure 7.3 Convection boundary condition at node $M+1$ of an M-element, one-dimensional heat transfer finite element model.

7.3.2 Boundary Conditions

In the one-dimensional case of heat transfer under consideration, two boundary conditions must be specified. Typically, this means that, if the finite element model of the problem is composed of M elements, one boundary condition is imposed at node 1 of element 1 and the second boundary condition is imposed at node 2 of element M. The boundary conditions are of three types:

1. *Imposed temperature*. The temperature at an end node is a known value; this condition occurs when an end of the body is subjected to a constant process temperature and heat is removed from the process by the body.
2. *Imposed heat flux*. The heat flow rate into, or out of, an end of the body is specified; while distinctly possible in a mathematical sense, this type of boundary condition is not often encountered in practice.
3. *Convection through an end node*. In this case, the end of the body is in contact with a fluid of known ambient temperature and the conduction flux at the boundary is removed via convection to the fluid media. Assuming that this condition applies at node 2 of element M of the finite element model, as in Figure 7.3, the convection boundary condition is expressed as

$$k_x \left.\frac{dT}{dx}\right|_{M+1} = -q_{M+1} = -h(T_{M+1} - T_a) \tag{7.19}$$

indicating that the conduction heat flux at the end node must be carried away by convection at that node. The area for convection in Equation 7.19 is the cross-sectional area of element M; as this area is common to each of the three terms in the equation, the area has been omitted. An explanation of the algebraic signs in Equation 7.19 is appropriate here. If $T_{M+1} > T_a$, the temperature gradient is negative (given the positive direction of the x axis as shown); therefore, the flux and convection are positive terms. The following example illustrates application of the one-dimensional conduction/convection problem.

EXAMPLE 7.3

Figure 7.4a depicts a cylindrical pin that is one of several in a small heat exchange device. The left end of the pin is subjected to a constant temperature of 180°F. The right end of the pin is in contact with a chilled water bath maintained at constant temperature of 40°F.

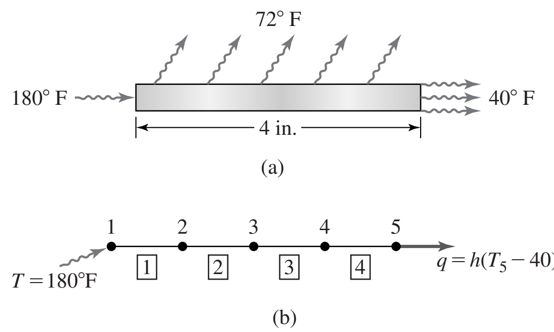

Figure 7.4 Example 7.3: (a) Cylindrical pin. (b) Finite element model.

The exterior surface of the pin is in contact with moving air at 72°F. The physical data are given as follows:

$$D = 0.5 \text{ in.},\qquad L = 4 \text{ in.},\qquad k_x = 120 \text{ Btu/(hr-ft-}^\circ\text{F)},$$
$$h_{\text{air}} = 50 \text{ Btu/(hr-ft}^2\text{-}^\circ\text{F)},\qquad h_{\text{water}} = 100 \text{ Btu/(hr-ft}^2\text{-}^\circ\text{F)}$$

Use four equal-length, two-node elements to obtain a finite element solution for the temperature distribution across the length of the pin and the heat flow rate through the pin.

■ **Solution**

Figure 7.4b shows the elements, node numbers, and boundary conditions. The boundary conditions are expressed as follows

At node 1: $$T_1 = 180^\circ\text{F}$$

At node 5: $$k_x \left.\frac{\mathrm{d}T}{\mathrm{d}x}\right|_5 = -q_5 = -h(T_5 - 40)$$

Element geometric data is then

$$L_e = 1 \text{ in.},\qquad P = \pi(0.5) = 1.5708 \text{ in.},\qquad A = (\pi/4)(0.5)^2 = 0.1963 \text{ in.}^2$$

The leading coefficients of the conductance matrix terms are

$$\frac{k_x A}{L_e} = \frac{120\left(\dfrac{0.1963}{144}\right)}{\dfrac{1}{12}} = 1.9630 \text{ Btu/(hr-}^\circ\text{F)}$$

$$\frac{h_{\text{air}} P L_e}{6} = \frac{50\left(\dfrac{1.5708}{12}\right)\left(\dfrac{1}{12}\right)}{6} = 0.0909 \text{ Btu/(hr-}^\circ\text{F)}$$

where conversion from inches to feet is to be noted.

Substituting into Equation 7.15, the element conductance matrix is

$$\left[k^{(e)}\right] = 1.9630\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + 0.0909\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2.1448 & -1.8721 \\ -1.8721 & 2.1448 \end{bmatrix}$$

Following the direct assembly procedure, the system conductance matrix is

$$[K] = \begin{bmatrix} 2.1448 & -1.8721 & 0 & 0 & 0 \\ -1.8721 & 4.2896 & -1.8721 & 0 & 0 \\ 0 & -1.8721 & 4.2896 & -1.8721 & 0 \\ 0 & 0 & -1.8721 & 4.2896 & -1.8721 \\ 0 & 0 & 0 & -1.8721 & 2.1448 \end{bmatrix}$$

As no internal heat is generated, $f_Q = 0$. The element convection force components per Equation 7.18 are

$$\left\{f_h^{(e)}\right\} = \frac{hPT_aL}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \frac{50\left(\dfrac{1.5708}{12}\right)(72)\left(\dfrac{1}{12}\right)}{2} = \begin{Bmatrix} 19.6375 \\ 19.6375 \end{Bmatrix} \text{ Btu/hr}$$

Assembling the contributions of each element at the nodes gives the system convection force vector as

$$\{F_h\} = \begin{Bmatrix} 19.6375 \\ 39.2750 \\ 39.2750 \\ 39.2750 \\ 19.6375 \end{Bmatrix} \text{ Btu/hr}$$

Noting the cancellation of terms at nodal connections, the system gradient vector becomes simply

$$\{F_g\} = \begin{Bmatrix} Aq_1 \\ 0 \\ 0 \\ 0 \\ -Aq_5 \end{Bmatrix} = \begin{Bmatrix} Aq_1 \\ 0 \\ 0 \\ 0 \\ -Ah_{\text{water}}(T_5 - 40) \end{Bmatrix} = \begin{Bmatrix} Aq_1 \\ 0 \\ 0 \\ 0 \\ -0.1364T_5 + 5.4542 \end{Bmatrix} \text{ Btu/hr}$$

and the boundary condition at the pin-water interface has been explicitly incorporated. Note that, as a result of the convection boundary condition, a term containing unknown nodal temperature T_5 appears in the gradient vector. This term is transposed in the final equations and results in a increase in value of the K_{55} term of the system matrix. The final assembled equations are

$$\begin{bmatrix} 2.1448 & -1.8721 & 0 & 0 & 0 \\ -1.8721 & 4.2896 & -1.8721 & 0 & 0 \\ 0 & -1.8721 & 4.2896 & -1.8721 & 0 \\ 0 & 0 & -1.8721 & 4.2896 & -1.8721 \\ 0 & 0 & 0 & -1.8721 & 2.2812 \end{bmatrix}\begin{Bmatrix} 180 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 19.6375 + Aq_1 \\ 39.2750 \\ 39.2750 \\ 39.2750 \\ 25.0917 \end{Bmatrix}$$

Eliminating the first equation while taking care to include the effect of the specified temperature at node 1 on the remaining equations gives

$$\begin{bmatrix} 4.2896 & -1.8571 & 0 & 0 \\ -1.8721 & 4.2896 & -1.8721 & 0 \\ 0 & -1.8721 & 4.2896 & -1.8721 \\ 0 & 0 & -1.8721 & 2.2812 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 376.2530 \\ 39.2750 \\ 39.2750 \\ 25.0917 \end{Bmatrix}$$

Solving by Gaussian elimination, the nodal temperatures are obtained as

$$T_2 = 136.16°\text{F}$$
$$T_3 = 111.02°\text{F}$$
$$T_4 = 97.23°\text{F}$$
$$T_5 = 90.79°\text{F}$$

The heat flux at node 1 is computed by back substitution of T_2 into the first equation:

$$2.1448(180) - 1.8721(136.16) = 19.6375 + Aq_1$$
$$Aq_1 = 111.5156 \text{ Btu/hr}$$
$$q_1 = \frac{111.5156}{0.1963/144} \approx 81{,}805 \text{ Btu/hr-ft}^2$$

Although the pin length in this example is quite small, use of only four elements represents a coarse element mesh. To illustrate the effect, recall that, for the linear, two-node element the first derivative of the field variable, in this case, the temperature gradient, is constant; that is,

$$\frac{\mathrm{d}T}{\mathrm{d}x} = \frac{\Delta T}{\Delta x} = \frac{\Delta T}{L_e}$$

Using the computed nodal temperatures, the element gradients are

Element 1: $$\frac{\mathrm{d}T}{\mathrm{d}x} = \frac{136.16 - 180}{1} = -43.84 \frac{°\text{F}}{\text{in.}}$$

Element 2: $$\frac{\mathrm{d}T}{\mathrm{d}x} = \frac{111.02 - 136.16}{1} = -25.14 \frac{°\text{F}}{\text{in.}}$$

Element 3: $$\frac{\mathrm{d}T}{\mathrm{d}x} = \frac{97.23 - 111.02}{1} = -13.79 \frac{°\text{F}}{\text{in.}}$$

Element 4: $$\frac{\mathrm{d}T}{\mathrm{d}x} = \frac{90.79 - 97.23}{1} = -6.44 \frac{°\text{F}}{\text{in.}}$$

where the length is expressed in inches for numerical convenience. The computed gradient values show significant discontinuities at the nodal connections. As the number of elements is increased, the magnitude of such jump discontinuities in the gradient values decrease significantly as the finite element approximation approaches the true solution.

Table 7.1 Nodal Temperature Solutions

x (inches)	Four Elements, *T* (°F)	Eight Elements, *T* (°F)
0	180	180
0.5	158.08*	155.31
1.0	136.16	136.48
1.5	123.59*	122.19
2.0	111.02	111.41
2.5	104.13*	103.41
3.0	97.23	97.62
3.5	94.01*	93.63
4.0	90.79	91.16

To illustrate convergence as well as the effect on gradient values, an eight-element solution was obtained for this problem. Table 7.1 shows the nodal temperature solutions for both four- and eight-element models. Note that, in the table, values indicated by * are interpolated, nonnodal values.

7.4 HEAT TRANSFER IN TWO DIMENSIONS

A case in which heat transfer can be considered to be adequately described by a two-dimensional formulation is shown in Figure 7.5. The rectangular fin has dimensions $a \times b \times t$, and thickness t is assumed small in comparison to a and b. One edge of the fin is subjected to a known temperature while the other three edges and the faces of the fin are in contact with a fluid. Heat transfer then occurs from the core via conduction through the fin to its edges and faces, where convection takes place. The situation depicted could represent a cooling fin removing heat from some process or a heating fin moving heat from an energy source to a building space.

To develop the governing equations, we refer to a differential element of a solid body that has a small dimension in the z direction, as in Figure 7.6, and examine the principle of conservation of energy for the differential element. As we now deal with two dimensions, all derivatives are partial derivatives. Again, on the edges $x + \mathrm{d}x$ and $y + \mathrm{d}y$, the heat flux terms have been expanded in first-order Taylor series. We assume that the differential element depicted is in the interior of the body, so that convection occurs only at the surfaces of the element and not along the edges. Applying Equation 5.53 under the assumption of steady-state conditions (i.e., $\Delta U = 0$), we obtain

$$q_x t\,\mathrm{d}y + q_y t\,\mathrm{d}x + Qt\,\mathrm{d}y\,\mathrm{d}x = \left(q_x + \frac{\partial q_x}{\partial x}\mathrm{d}x\right)t\,\mathrm{d}y + \left(q_y + \frac{\partial q_y}{\partial y}\mathrm{d}y\right)t\,\mathrm{d}x + 2h(T - T_a)\,\mathrm{d}y\,\mathrm{d}x \tag{7.20}$$

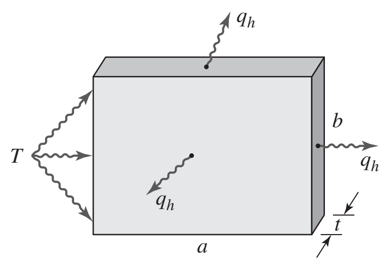

Figure 7.5 Two-dimensional conduction fin with face and edge convection.

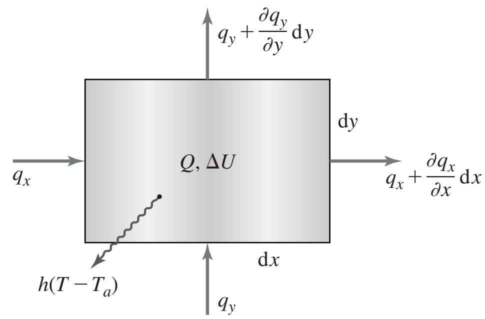

Figure 7.6 Differential element depicting two-dimensional conduction with surface convection.

where

t = thickness

h = the convection coefficient from the surfaces of the differential element

T_a = the ambient temperature of the surrounding fluid

Utilizing Fourier's law in the coordinate directions

$$q_x = -k_x \frac{\partial T}{\partial x}$$
$$q_y = -k_y \frac{\partial T}{\partial y} \tag{7.21}$$

then substituting and simplifying yields

$$Qt\,\mathrm{d}y\,\mathrm{d}x = \frac{\partial}{\partial x}\left(-k_x \frac{\partial T}{\partial x}\right) t\,\mathrm{d}y\,\mathrm{d}x + \frac{\partial}{\partial y}\left(-k_y \frac{\partial T}{\partial y}\right) t\,\mathrm{d}y\,\mathrm{d}x + 2h(T - T_a)\,\mathrm{d}y\,\mathrm{d}x \tag{7.22}$$

where k_x and k_y are the thermal conductivities in the x and y directions, respectively. Equation 7.22 simplifies to

$$\frac{\partial}{\partial x}\left(tk_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(tk_y \frac{\partial T}{\partial y}\right) + Qt = 2h(T - T_a) \tag{7.23}$$

Equation 7.23 is the governing equation for two-dimensional conduction with convection from the surfaces of the body. Convection from the edges is also possible, as is subsequently discussed in terms of the boundary conditions.

7.4.1 Finite Element Formulation

In developing a finite element approach to two-dimensional conduction with convection, we take a general approach initially; that is, a specific element geometry is not used. Instead, we assume a two-dimensional element having M nodes such that the temperature distribution in the element is described by

$$T(x, y) = \sum_{i=1}^{M} N_i(x, y)T_i = [N]\{T\} \tag{7.24}$$

where $N_i(x, y)$ is the interpolation function associated with nodal temperature T_i, $[N]$ is the row matrix of interpolation functions, and $\{T\}$ is the column matrix (vector) of nodal temperatures.

Applying Galerkin's finite element method, the residual equations corresponding to Equation 7.23 are

$$\iint_A N_i(x, y)\left[\frac{\partial}{\partial x}\left(tk_x\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(tk_y\frac{\partial T}{\partial y}\right) + Qt - 2h(T - T_a)\right] \mathrm{d}A = 0 \qquad i = 1, M \tag{7.25}$$

where thickness t is assumed constant and the integration is over the area of the element. (Strictly speaking, the integration is over the volume of the element, since the volume is the domain of interest.) To develop the finite element equations for the two-dimensional case, a bit of mathematical manipulation is required.

Consider the first two integrals in Equation 7.25 as

$$t\iint_A \left[\frac{\partial}{\partial x}\left(k_x\frac{\partial T}{\partial x}\right)N_i + \frac{\partial}{\partial y}\left(k_y\frac{\partial T}{\partial y}\right)N_i\right] \mathrm{d}A = -t\iint_A \left(\frac{\partial q_x}{\partial x}N_i + \frac{\partial q_y}{\partial y}N_i\right) \mathrm{d}A \tag{7.26}$$

and note that we have used Fourier's law per Equation 7.21. For illustration, we now assume a rectangular element, as shown in Figure 7.7a, and examine

$$t\iint_A \frac{\partial q_x}{\partial x}N_i \,\mathrm{d}A = t\int_{y_1}^{y_2}\int_{x_1}^{x_2} \frac{\partial q_x}{\partial x}N_i \,\mathrm{d}x\,\mathrm{d}y \tag{7.27}$$

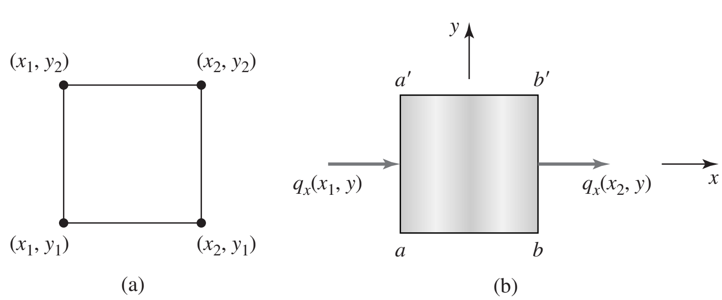

Figure 7.7 Illustration of boundary heat flux in x direction.

Integrating by parts on x with $u = N_i$ and $\mathrm{d}v = \dfrac{\partial q_x}{\partial x}\,\mathrm{d}x$, we obtain, formally,

$$t\iint\limits_A \frac{\partial q_x}{\partial x} N_i\,\mathrm{d}A = t\int\limits_{y_1}^{y_2} q_x N_i\Big|_{x_1}^{x_2}\mathrm{d}y - t\int\limits_{y_1}^{y_2}\int\limits_{x_1}^{x_2} q_x \frac{\partial N_i}{\partial x}\,\mathrm{d}x\,\mathrm{d}y$$

$$= t\int\limits_{y_1}^{y_2} q_x N_i\Big|_{x_1}^{x_2}\mathrm{d}y + t\int\limits_A k_x \frac{\partial T}{\partial x}\frac{\partial N_i}{\partial x}\,\mathrm{d}A \tag{7.28}$$

Now let us examine the physical significance of the term

$$t\int\limits_{y_1}^{y_2} q_x N_i\Big|_{x_1}^{x_2}\mathrm{d}y = t\int\limits_{y_1}^{y_2}[q_x(x_2, y)N_i(x_2, y) - q_x(x_1, y)N_i(x_1, y)]\,\mathrm{d}y \tag{7.29}$$

The integrand is the weighted value (N_i is the scalar weighting function) of the heat flux in the x direction across edges a-a' and b-b' in Figure 7.7b. Hence, when we integrate on y, we obtain the *difference* in the weighted heat flow rate in the x direction across b-b' and a-a', respectively. Noting the obvious fact that the heat flow rate in the x direction across horizontal boundaries a-b and a'-b' is zero, the integral over the area of the element is equivalent to an integral around the periphery of the element, as given by

$$t\iint\limits_A q_x N_i\,\mathrm{d}A = t\oint\limits_S q_x N_i n_x\,\mathrm{d}S \tag{7.30}$$

In Equation 7.30, S is the periphery of the element and n_x is the x component of the *outward* unit vector normal (perpendicular) to the periphery. In our example, using a rectangular element, we have $n_x = 1$ along b-b', $n_x = 0$ along b'-a', $n_x = -1$ along a'-a, and $n_x = 0$ along a-b. Note that the use of the normal vector component ensures that the directional nature of the heat flow is accounted for properly. For theoretical reasons beyond the scope of this text, the integration around the periphery S is to be taken in the counterclockwise direction; that is, positively, per the right-hand rule.

An identical argument and development will show that, for the y-direction terms in equation Equation 7.26,

$$t\iint\limits_A \frac{\partial}{\partial y}\left(k_y\frac{\partial T}{\partial y}\right)N_i\,\mathrm{d}A = -t\oint\limits_S q_y N_i n_y\,\mathrm{d}S - \int\limits_A k_y\frac{\partial T}{\partial y}\frac{\partial N_i}{\partial y}\,\mathrm{d}A \tag{7.31}$$

These arguments, based on the specific case of a rectangular element, are intended to show an application of a general relation known as the *Green-Gauss theorem* (also known as *Green's theorem in the plane*) stated as follows: *Let $F(x, y)$ and $G(x, y)$ be continuous functions defined in a region of the x-y plane* (for our purposes the region is the area of an element); *then*

$$\iint_A \left(\beta\frac{\partial F}{\partial x}+\beta\frac{\partial G}{\partial y}\right) = \oint_S (\beta F n_x + \beta G n_y)\,\mathrm{d}S - \iint_A \left(\frac{\partial F}{\partial x}\frac{\partial \beta}{\partial x}+\frac{\partial G}{\partial y}\frac{\partial \beta}{\partial y}\right)\mathrm{d}A \tag{7.32}$$

Returning to Equation 7.26, we let $F = k_x\frac{\partial T}{\partial x}$, $G = k_y\frac{\partial T}{\partial y}$, and $\beta = N_i(x, y)$, and apply the Green-Gauss theorem to obtain

$$t\iint_A \left[\frac{\partial}{\partial x}\left(k_x\frac{\partial T}{\partial x}\right)N_i + \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}\right)N_i\right]\mathrm{d}A = -t\oint_S (q_x n_x + q_y n_y)N_i\,\mathrm{d}S - t\iint_A \left(k_x\frac{\partial T}{\partial x}\frac{\partial N_i}{\partial x}+k_y\frac{\partial T}{\partial y}\frac{\partial N_i}{\partial y}\right)\mathrm{d}A \tag{7.33}$$

Application of the Green-Gauss theorem, as in this development, is the two-dimensional counterpart of integration by parts in one dimension. The result is that we have introduced the boundary gradient terms as indicated by the first integral on the right-hand side of Equation 7.33 and ensured that the conductance matrix is symmetric, per the second integral, as will be seen in the remainder of the development.

Returning to the Galerkin residual equation represented by Equation 7.25 and substituting the relations developed via the Green-Gauss theorem (being careful to observe arithmetic signs), Equation 7.25 becomes

$$\iint_A \left(k_x\frac{\partial T}{\partial x}\frac{\partial N_i}{\partial x}+k_y\frac{\partial T}{\partial y}\frac{\partial N_i}{\partial y}\right)t\,\mathrm{d}A + 2h\iint_A T N_i\,\mathrm{d}A = \iint_A QN_i t\,\mathrm{d}A + 2hT_a\iint_A N_i\,\mathrm{d}A - t\oint_S (q_x n_x + q_y n_y)N_i\,\mathrm{d}S \qquad i = 1, M \tag{7.34}$$

as the system of *M* equations for the two-dimensional finite element formulation via Galerkin's method. In analogy with the one-dimensional case of Equation 7.8, we observe that the left-hand side includes the unknown temperature distribution while the right-hand side is composed of forcing functions, representing internal heat generation, surface convection, and boundary heat flux.

At this point, we convert to matrix notation for ease of illustration by employing Equation 7.24 to convert Equation 7.34 to

$$\iint_A \left(k_x\left[\frac{\partial N}{\partial x}\right]^T\left[\frac{\partial N}{\partial x}\right]+k_y\left[\frac{\partial N}{\partial y}\right]^T\left[\frac{\partial N}{\partial y}\right]\right)\{T\}t\,\mathrm{d}A + 2h\iint_A [N]^T[N]\{T\}\,\mathrm{d}A = \iint_A Q[N]^T t\,\mathrm{d}A + 2hT_a\iint_A [N]^T\,\mathrm{d}A - \oint_S q_s n_s[N]^T t\,\mathrm{d}S \tag{7.35}$$

which is of the form

$$\left[k^{(e)}\right]\{T\} = \left\{f_Q^{(e)}\right\} + \left\{f_h^{(e)}\right\} + \left\{f_g^{(e)}\right\} \tag{7.36}$$

as desired.

Comparison of Equations 7.35 and 7.36 shows that the conductance matrix is

$$\left[k^{(e)}\right] = \iint_A \left(k_x \left[\frac{\partial N}{\partial x}\right]^T \left[\frac{\partial N}{\partial x}\right] + k_y \left[\frac{\partial N}{\partial y}\right]^T \left[\frac{\partial N}{\partial y}\right]\right) t \, \mathrm{d}A + 2h \iint_A [N]^T [N] \, \mathrm{d}A \tag{7.37}$$

which for an element having M nodes is an $M \times M$ symmetric matrix. While we use the term *conductance matrix,* the first integral term on the right of Equation 7.37 represents the conduction "stiffness," while the second integral represents convection from the lateral surfaces of the element to the surroundings. If the lateral surfaces do not exhibit convection (i.e., the surfaces are insulated), the convection terms are removed by setting $h = 0$. Note that, in many finite element software packages, the convection portion of the conductance matrix is not automatically included in element matrix formulation. Instead, lateral surface (as well as edge) convection effects are specified by applying convection "loads" to the surfaces as appropriate. The software then modifies the element matrices as required.

The element forcing functions are described in column matrix (vector) form as

$$\begin{aligned}
\left\{f_Q^{(e)}\right\} &= \iint_A Q[N]^T t \, \mathrm{d}A = \iint_A Q\{N\} t \, \mathrm{d}A \\
\left\{f_h^{(e)}\right\} &= 2hT_a \iint_A [N]^T \, \mathrm{d}A = 2hT_a \iint_A \{N\} \, \mathrm{d}A \\
\left\{f_g^{(e)}\right\} &= -\oint_S q_s n_s [N]^T t \, \mathrm{d}S = -\oint_S q_s n_s \{N\} t \, \mathrm{d}S
\end{aligned} \tag{7.38}$$

where $[N]^T = \{N\}$ is the $M \times 1$ column matrix of interpolation functions.

Equations 7.36–7.38 represent the general formulation of a finite element for two-dimensional heat conduction with convection from the surfaces. Note in particular that these equations are valid for an arbitrary element having M nodes and, therefore, any order of interpolation functions (linear, quadratic, cubic, etc.). In following examples, use of specific element geometries are illustrated.

7.4.2 Boundary Conditions

The boundary conditions for two-dimensional conduction with convection may be of three types, as illustrated by Figure 7.8 for a general two-dimensional domain. On portion S_1 of the boundary, the temperature is prescribed as a known

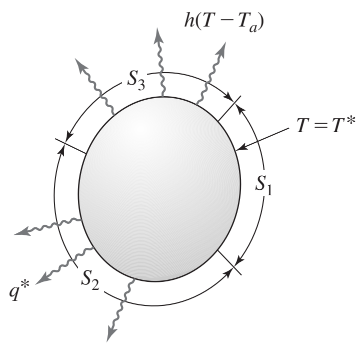

Figure 7.8 Types of boundary conditions for two-dimensional conduction with convection.

constant value $T_{S1} = T^*$. In a finite element model of such a domain, every element node located on S_1 has known temperature and the corresponding nodal equilibrium equations become "reaction" equations. The reaction "forces" are the heat fluxes at the nodes on S_1. In using finite element software packages, such conditions are input data; the user of the software ("FE programmer") enters such data as appropriate at the applicable nodes of the finite element model (in this case, specified temperatures).

The heat flux on portion S_2 of the boundary is prescribed as $q_{S2} = q^*$. This is analogous to specified nodal forces in a structural problem. Hence, for all elements having nodes on S_2, the third of Equation 7.38 gives the corresponding nodal forcing functions as

$$\left\{ f_g^{(e)} \right\} = -\oint_{S_2} q^* n_{S_2} \{N\} t \, \mathrm{d}S \tag{7.39}$$

Finally, a portion S_3 of the boundary illustrates an edge convection condition. In this situation, the heat flux at the boundary must be equilibrated by the convection loss from S_3. For all elements having edges on S_3, the convection condition is expressed as

$$\left\{ f_g^{(e)} \right\} = -\oint_{S_3} q_{S_3} n_{S_3} \{N\} t \, \mathrm{d}S = -\oint_{S_3} h(T^{(e)} - T_a) \{N\} t \, \mathrm{d}S \tag{7.40}$$

Noting that the right-hand side of Equation 7.40 involves the nodal temperatures, we rewrite the equation as

$$\left\{ f_g^{(e)} \right\} = -\oint_{S_3} h[N]^T [N] \{T\} t \, \mathrm{d}S_3 + \oint_{S_3} h T_a \{N\} t \, \mathrm{d}S_3 \tag{7.41}$$

and observe that, when inserted into Equation 7.36, the first integral term on the right of Equation 7.41 adds stiffness to specific terms of the conductance matrix associated with nodes on S_3. To generalize, we rewrite Equation 7.41 as

$$\left\{f_g^{(e)}\right\} = -\left[k_{hS}^{(e)}\right]\{T\} + \left\{f_{hS}^{(e)}\right\} \tag{7.42}$$

where

$$\left[k_{hS}^{(e)}\right] = \oint_S h[N]^T[N]t \, \mathrm{d}S \tag{7.43}$$

is the contribution to the element conductance matrix owing to convection on portion S of the element boundary and

$$\left\{f_{hS}^{(e)}\right\} = \oint_S hT_a\{N\}t \, \mathrm{d}S \tag{7.44}$$

is the forcing function associated with convection on S.

Incorporating Equation 7.42 into Equation 7.36, we have

$$\left[k^{(e)}\right]\{T\} = \left\{f_Q^{(e)}\right\} + \left\{f_h^{(e)}\right\} + \left\{f_g^{(e)}\right\} + \left\{f_{hS}^{(e)}\right\} \tag{7.45}$$

where the element conductance matrix is now given by

$$\left[k^{(e)}\right] = \iint_A \left(k_x \left[\frac{\partial N}{\partial x}\right]^T \left[\frac{\partial N}{\partial x}\right] + k_y \left[\frac{\partial N}{\partial y}\right]^T \left[\frac{\partial N}{\partial y}\right]\right) t \, \mathrm{d}A + 2h \iint_A [N]^T [N] \, \mathrm{d}A + h \oint_S [N]^T [N] \, t \, \mathrm{d}S \tag{7.46}$$

which now explicitly includes edge convection on portion(s) S of the element boundary subjected to convection.

EXAMPLE 7.4

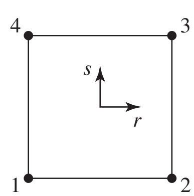

Figure 7.9 Element node numbering for Example 7.4; the length of each edge is 1 in.

Determine the conductance matrix (excluding edge convection) for a four-node, rectangular element having 0.5 in. thickness and equal sides of 1 in. The material has thermal properties $k_x = k_y = 20$ Btu/(hr-ft-°F) and $h = 50$ Btu/(hr-ft^2-°F).

■ Solution

The element with node numbers is as shown in Figure 7.9 and the interpolation functions, Equation 6.56, are

$$N_1(r, s) = \frac{1}{4}(1 - r)(1 - s)$$

$$N_2(r, s) = \frac{1}{4}(1 + r)(1 - s)$$

$$N_3(r,s) = \frac{1}{4}(1+r)(1+s)$$

$$N_4(r,s) = \frac{1}{4}(1-r)(1+s)$$

in terms of the normalized coordinates r and s. For the 1-in. square element, we have $2a = 2b = 1$ and $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y = ab\,\mathrm{d}r\,\mathrm{d}s$. The partial derivatives in terms of the normalized coordinates, via the chain rule, are

$$\frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial r}\frac{\partial r}{\partial x} = \frac{1}{a}\frac{\partial N_i}{\partial r} \qquad i = 1, 4$$

$$\frac{\partial N_i}{\partial y} = \frac{\partial N_i}{\partial s}\frac{\partial s}{\partial y} = \frac{1}{b}\frac{\partial N_i}{\partial s} \qquad i = 1, 4$$

Therefore, Equation 7.37 becomes

$$\left[k^{(e)}\right] = \int_{-1}^{1}\int_{-1}^{1}\left(k_x\left[\frac{\partial N}{\partial r}\right]^T\left[\frac{\partial N}{\partial r}\right]\frac{1}{a^2} + k_y\left[\frac{\partial N}{\partial s}\right]^T\left[\frac{\partial N}{\partial s}\right]\frac{1}{b^2}\right)tab\,\mathrm{d}r\,\mathrm{d}s$$
$$+ 2h\int_{-1}^{1}[N]^T[N]ab\,\mathrm{d}r\,\mathrm{d}s$$

or, on a term by term basis,

$$k_{ij} = \int_{-1}^{1}\int_{-1}^{1}\left(k_x\frac{\partial N_i}{\partial r}\frac{\partial N_j}{\partial r}\frac{1}{a^2} + k_y\frac{\partial N_i}{\partial s}\frac{\partial N_j}{\partial s}\frac{1}{b^2}\right)tab\,\mathrm{d}r\,\mathrm{d}s$$
$$+ 2h\int_{-1}^{1}\int_{-1}^{1}N_iN_jab\,\mathrm{d}r\,\mathrm{d}s \qquad i, j = 1, 4$$

or

$$k_{ij} = \int_{-1}^{1}\int_{-1}^{1}\left(k_x\frac{\partial N_i}{\partial r}\frac{\partial N_j}{\partial r}\frac{b}{a} + k_y\frac{\partial N_i}{\partial s}\frac{\partial N_j}{\partial s}\frac{a}{b}\right)t\,\mathrm{d}r\,\mathrm{d}s$$
$$+ 2h\int_{-1}^{1}\int_{-1}^{1}N_iN_jab\,\mathrm{d}r\,\mathrm{d}s \qquad i, j = 1, 4$$

Assuming that k_x and k_y are constants, we have

$$k_{ij} = k_xt\frac{b}{a}\int_{-1}^{1}\int_{-1}^{1}\frac{\partial N_i}{\partial r}\frac{\partial N_j}{\partial r}\,\mathrm{d}r\,\mathrm{d}s + k_yt\frac{a}{b}\int_{-1}^{1}\int_{-1}^{1}\frac{\partial N_i}{\partial s}\frac{\partial N_j}{\partial s}\,\mathrm{d}r\,\mathrm{d}s$$
$$+ 2\,hab\int_{-1}^{1}\int_{-1}^{1}N_iN_j\,\mathrm{d}r\,\mathrm{d}s \qquad i, j = 1, 4$$

The required partial derivatives are

$$\frac{\partial N_1}{\partial r} = \frac{1}{4}(s-1) \qquad \frac{\partial N_1}{\partial s} = \frac{1}{4}(r-1)$$

$$\frac{\partial N_2}{\partial r} = \frac{1}{4}(1-s) \qquad \frac{\partial N_2}{\partial s} = -\frac{1}{4}(1+r)$$

$$\frac{\partial N_3}{\partial r} = \frac{1}{4}(1+s) \qquad \frac{\partial N_3}{\partial s} = \frac{1}{4}(1+r)$$

$$\frac{\partial N_4}{\partial r} = -\frac{1}{4}(1+s) \qquad \frac{\partial N_4}{\partial s} = \frac{1}{4}(1-r)$$

Substituting numerical values (noting that $a = b$), we obtain, for example,

$$k_{11} = 20\int_{-1}^{1}\int_{-1}^{1}\left[\frac{1}{16}(s-1)^2 + \frac{1}{16}(r-1)^2\right]\left(\frac{0.5}{12}\right)\text{d}r\,\text{d}s$$

$$+\,2(50)\int_{-1}^{1}\int_{-1}^{1}\frac{1}{16}(1-r)^2(1-s)^2\left(\frac{0.5}{12}\right)^2\text{d}r\,\text{d}s$$

Integrating first on r,

$$k_{11} = \frac{20(0.5)}{16(12)}\int_{-1}^{1}(s-1)^2\,r\Big|_{-1}^{1} + \left.\frac{(r-1)^3}{3}\right|_{-1}^{1}\text{d}s$$

$$-\,\frac{100}{16}\left(\frac{0.5}{12}\right)^2\int_{-1}^{1}(1-s)^2\left.\frac{(1-r)^3}{3}\right|_{-1}^{1}\text{d}s$$

or

$$k_{11} = \frac{20(0.5)}{16(12)}\int_{-1}^{1}\left[(s-1)^2(2)+\frac{8}{3}\right]\text{d}s + \frac{100}{16}\left(\frac{0.5}{12}\right)^2\int_{-1}^{1}(1-s)^2\frac{8}{3}\,\text{d}s$$

Then, integrating on s, we obtain

$$k_{11} = \frac{20(0.5)}{16(12)}\left(\frac{2(s-1)^3}{3}+\frac{8}{3}s\right)\Bigg|_{-1}^{1} - \frac{100}{16}\left(\frac{0.5}{12}\right)^2\left(\frac{8}{3}\right)\left(\frac{(1-s)^3}{3}\right)_{-1}^{1}$$

or

$$k_{11} = \frac{20(0.5)}{16(12)}\left(\frac{16}{3}+\frac{16}{3}\right) + \frac{100}{16}\left(\frac{0.5}{12}\right)^2\left(\frac{8}{3}\right)\left(\frac{8}{3}\right) = 0.6327\ \text{Btu/(hr-}^\circ\text{F)}$$

The analytical integration procedure just used to determine k_{11} is *not* the method used by finite element software packages; instead, numerical methods are used, primarily the Gauss quadrature procedure discussed in Chapter 6. If we examine the terms in the integrands of the equation defining k_{ij}, we find that the integrands are quadratic functions of r and s. Therefore, the integrals can be evaluated exactly by using two Gauss points in r and s. Per Table 6.1, the required Gauss points and weighting factors are r_i, $s_j = \pm\, 0.57735$ and $W_i, W_j = 1.0, i, j = 1, 2$. Using the numerical procedure for k_{11}, we write

$$k_{11} = k_x t \frac{b}{a} \int_{-1}^{1}\int_{-1}^{1} \frac{1}{16}(s-1)^2 \,\mathrm{d}r \,\mathrm{d}s + k_y t \frac{b}{a} \int_{-1}^{1}\int_{-1}^{1} \frac{1}{16}(r-1)^2 \,\mathrm{d}r \,\mathrm{d}s$$

$$+ 2hab \int_{-1}^{1} \frac{1}{16}(r-1)^2(s-1)^2 \,\mathrm{d}r \,\mathrm{d}s$$

$$= k_x t \frac{b}{a} \sum_{i=1}^{2}\sum_{j=1}^{2} \frac{1}{16} W_i W_j (s_j - 1)^2 + k_y t \frac{a}{b} \sum_{i=1}^{2}\sum_{j=1}^{2} \frac{1}{16} W_i W_j (r_i - 1)^2$$

$$+ 2hab \sum_{i=1}^{2}\sum_{j=1}^{2} \frac{1}{16} W_i W_j (1 - r_i)^2 (1 - s_j)^2$$

and, using the specified integration points and weighting factors, this evaluates to

$$k_{11} = k_x t \frac{b}{a}\left(\frac{1}{3}\right) + k_y t \frac{a}{b}\left(\frac{1}{3}\right) + 2hab\left(\frac{4}{9}\right)$$

It is extremely important to note that the result expressed in the preceding equation is the correct value of k_{11} for *any* rectangular element used for the two-dimensional heat conduction analysis discussed in this section. The integrations need not be repeated for each element; only the geometric quantities and the conductance values need be substituted to obtain the value. Indeed, if we substitute the values for this example, we obtain

$$k_{11} = 0.6327 \text{ Btu/(hr-}^\circ\text{F)}$$

as per the analytical integration procedure.

Proceeding with the Gaussian integration procedure (calculation of some of these terms are to be evaluated as end-of-chapter problems), we find

$$k_{11} = k_{22} = k_{33} = k_{44} = 0.6327 \text{ Btu/(hr-}^\circ\text{F)}$$

Why are these values equal?

The off-diagonal terms (again using the numerical integration procedure) are calculated as

$$k_{12} = -0.1003$$
$$k_{13} = -0.2585$$
$$k_{14} = -0.1003$$
$$k_{23} = -0.1003$$
$$k_{24} = -0.2585$$
$$k_{34} = -0.1003$$

Btu/(hr-°F), and the complete element conductance matrix is

$$[k^{(e)}] = \begin{bmatrix} 0.6327 & -0.1003 & -0.2585 & -0.1003 \\ -0.1003 & 0.6327 & -0.1003 & -0.2585 \\ -0.2585 & -0.1003 & 0.6327 & -0.1003 \\ -0.1003 & -0.2585 & -0.1003 & 0.6327 \end{bmatrix} \text{Btu/(hr-°F)}$$

EXAMPLE 7.5

Figure 7.10a depicts a two-dimensional heating fin. The fin is attached to a pipe on its left edge, and the pipe conveys water at a constant temperature of 180°F. The fin is surrounded by air at temperature 68°F. The thermal properties of the fin are as given in Example 7.4. Use four equal-size four-node rectangular elements to obtain a finite element solution for the steady-state temperature distribution in the fin.

■ Solution

Figure 7.10b shows four elements with element and global node numbers. Given the numbering scheme selected, we have constant temperature conditions at global nodes 1, 2, and 3 such that

$$T_1 = T_2 = T_3 = 180°\text{F}$$

while on the other edges, we have convection boundary conditions that require a bit of analysis to apply. For element 1 (Figure 7.10c), for instance, convection occurs along element edge 1-2 but not along the other three element edges. Noting that $s = -1$ and

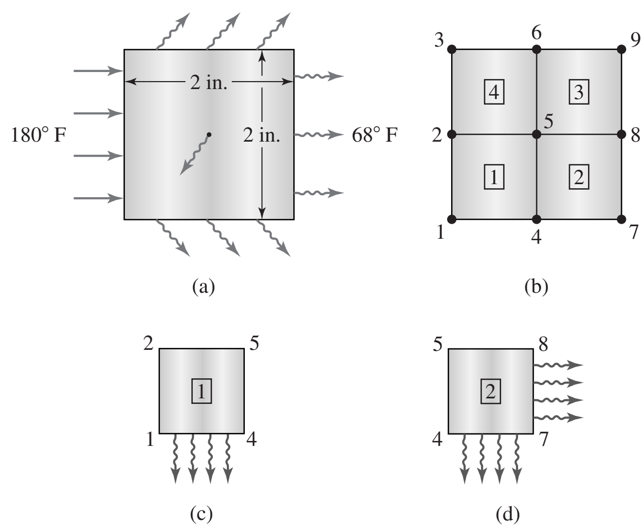

Figure 7.10 Example 7.5:
(a) Two-dimensional fin. (b) Finite element model.
(c) Element 1 edge convection. (d) Element 2 edge convection.

$N_3 = N_4 = 0$ on edge 1-2, Equation 7.43 becomes

$$\left[k_{hS}^{(1)}\right] = \left(\frac{1}{4}\right) ht \int_{-1}^{1} \begin{Bmatrix} 1-r \\ 1+r \\ 0 \\ 0 \end{Bmatrix} [\,1-r \quad 1+r \quad 0 \quad 0\,]\, a\, \mathrm{d}r$$

$$= \frac{hta}{4} \int_{-1}^{1} \begin{bmatrix} (1-r)^2 & 1-r^2 & 0 & 0 \\ 1-r^2 & (1+r)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathrm{d}r$$

Integrating as indicated gives

$$\left[k_{hS}^{(1)}\right] = \frac{hta}{4(3)} \begin{bmatrix} 8 & 4 & 0 & 0 \\ 4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \frac{50(0.5)^2}{4(3)(12)^2} \begin{bmatrix} 8 & 4 & 0 & 0 \\ 4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.0579 & 0.0290 & 0 & 0 \\ 0.0290 & 0.0579 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where the units are Btu/(hr-°F).

The edge convection force vector for element 1 is, per Equation 7.44,

$$\left\{f_{hS}^{(1)}\right\} = \frac{hT_a t}{2} \int_{-1}^{1} \begin{Bmatrix} 1-r \\ 1+r \\ 0 \\ 0 \end{Bmatrix} a\, \mathrm{d}r = \frac{hT_a ta}{2} \begin{Bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{Bmatrix} = \frac{50(68)(0.5)^2}{2(12)^2} \begin{Bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} 5.9028 \\ 5.9028 \\ 0 \\ 0 \end{Bmatrix} \text{Btu/hr}$$

where we again utilize $s = -1$, $N_3 = N_4 = 0$ along the element edge bounded by nodes 1 and 2.

Next consider element 2. As depicted in Figure 7.10d, convection occurs along two element edges defined by element nodes 1-2 ($s = -1$) and element nodes 2-3 ($r = 1$). For element 2, Equation 7.43 is

$$\left[k_{hS}^{(2)}\right] = \frac{ht}{4} \left(\int_{-1}^{1} \begin{Bmatrix} 1-r \\ 1+r \\ 0 \\ 0 \end{Bmatrix} [\,1-r \quad 1+r \quad 0 \quad 0\,] a\, \mathrm{d}r \right.$$

$$\left. + \int_{-1}^{1} \begin{Bmatrix} 0 \\ 1-s \\ 1+s \\ 0 \end{Bmatrix} [\,0 \quad 1-s \quad 1+s \quad 0\,] b\, \mathrm{d}s \right)$$

or, after integrating,

$$\left[k_{hS}^{(2)}\right] = \frac{hta}{4(3)}\begin{bmatrix} 8 & 4 & 0 & 0 \\ 4 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{htb}{4(3)}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 8 & 4 & 0 \\ 0 & 4 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and, since $a = b$,

$$\left[k_{hS}^{(2)}\right] = \frac{50(0.5)^2}{4(3)(12)^2}\begin{bmatrix} 8 & 4 & 0 & 0 \\ 4 & 16 & 4 & 0 \\ 0 & 4 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.0579 & 0.0290 & 0 & 0 \\ 0.0290 & 0.1157 & 0.0290 & 0 \\ 0 & 0.0290 & 0.0579 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{Btu/(hr-}^\circ\text{F)}$$

Likewise, the element edge convection force vector is obtained by integration along the two edges as

$$\left\{f_{hS}^{(2)}\right\} = \frac{hT_a t}{2}\left(\int_{-1}^{1}\begin{Bmatrix} 1-r \\ 1+r \\ 0 \\ 0 \end{Bmatrix} a\,\mathrm{d}r + \int_{-1}^{1}\begin{Bmatrix} 0 \\ 1-s \\ 1+s \\ 0 \end{Bmatrix} b\,\mathrm{d}s\right)$$

$$= \frac{50(68)(0.5)^2}{2(12)^2}\begin{Bmatrix} 2 \\ 4 \\ 2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 5.9028 \\ 11.8056 \\ 5.9028 \\ 0 \end{Bmatrix} \text{Btu/hr}$$

Identical procedures applied to the appropriate edges of elements 3 and 4 result in

$$\left[k_{hS}^{(3)}\right] = \frac{50(0.5)^2}{4(3)(12)^2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 8 & 4 & 0 \\ 0 & 4 & 16 & 4 \\ 0 & 0 & 4 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.0579 & 0.0290 & 0 \\ 0 & 0.0290 & 0.1157 & 0.0290 \\ 0 & 0 & 0.0290 & 0.0579 \end{bmatrix} \text{Btu/(hr-}^\circ\text{F)}$$

$$\left[k_{hS}^{(4)}\right] = \frac{50(0.5)^2}{4(3)(12)^2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 4 \\ 0 & 0 & 4 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0579 & 0.0290 \\ 0 & 0 & 0.0290 & 0.0579 \end{bmatrix} \text{Btu/(hr-}^\circ\text{F)}$$

$$\left\{f_{hS}^{(3)}\right\} = \begin{Bmatrix} 0 \\ 5.9028 \\ 11.8056 \\ 5.9028 \end{Bmatrix} \text{Btu/hr}$$

$$\left\{f_{hS}^{(4)}\right\} = \begin{Bmatrix} 0 \\ 0 \\ 5.9028 \\ 5.9028 \end{Bmatrix} \text{Btu/hr}$$

As no internal heat is generated, the corresponding $\{f_Q^{(e)}\}$ force vector for each element is zero; that is,

$$\left\{f_Q^{(e)}\right\} = \iint_A Q\{N\}\,\mathrm{d}A = \{0\}$$

for each element.

On the other hand, each element exhibits convection from its surfaces, so the lateral convection force vector is

$$\left\{f_h^{(e)}\right\} = 2hT_a \iint_A \{N\}\,\mathrm{d}A = 2hT_a \int_{-1}^{1}\int_{-1}^{1} \left(\frac{1}{4}\right) \begin{Bmatrix} (1-r)(1-s) \\ (1+r)(1-s) \\ (1+r)(1+s) \\ (1-r)(1+s) \end{Bmatrix} ab\,\mathrm{d}r\,\mathrm{d}s$$

which evaluates to

$$\left\{f_h^{(e)}\right\} = \frac{2hT_a ab}{4}\begin{Bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{Bmatrix} = \frac{2(50)(68)(0.5)^2}{4(12)^2}\begin{Bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{Bmatrix} = \begin{Bmatrix} 11.8056 \\ 11.8056 \\ 11.8056 \\ 11.8056 \end{Bmatrix}$$

and we note that, since the element is square, the surface convection forces are distributed equally to each of the four element nodes.

The global equations for the four-element model can now be assembled by writing the element-to-global nodal correspondence relations as

$$\begin{aligned} \left[L^{(1)}\right] &= [1 \quad 4 \quad 5 \quad 2] \\ \left[L^{(2)}\right] &= [4 \quad 7 \quad 8 \quad 5] \\ \left[L^{(3)}\right] &= [5 \quad 8 \quad 9 \quad 6] \\ \left[L^{(4)}\right] &= [2 \quad 5 \quad 6 \quad 3] \end{aligned}$$

and adding the edge convection terms to obtain the element stiffness matrices as

$$\left[k^{(1)}\right] = \begin{bmatrix} 0.6906 & -0.0713 & -0.2585 & -0.1003 \\ -0.0713 & 0.6906 & -0.1003 & -0.2585 \\ -0.2585 & -0.1003 & 0.6327 & -0.1003 \\ -0.1003 & -0.2585 & -0.1003 & 0.6327 \end{bmatrix}$$

$$\left[k^{(2)}\right] = \begin{bmatrix} 0.6906 & -0.0713 & -0.2585 & -0.1003 \\ -0.0713 & 0.7484 & -0.0713 & -0.2585 \\ -0.2585 & -0.0713 & 0.6906 & -0.1003 \\ -0.1003 & -0.2585 & -0.1003 & 0.6327 \end{bmatrix}$$

$$\left[k^{(3)}\right] = \begin{bmatrix} 0.6327 & -0.1003 & -0.2585 & -0.1003 \\ -0.1003 & 0.6906 & -0.0713 & -0.2585 \\ -0.2585 & -0.0713 & 0.7484 & -0.0713 \\ -0.1003 & -0.2585 & -0.0713 & 0.6906 \end{bmatrix}$$

$$[k^{(4)}] = \begin{bmatrix} 0.6327 & -0.1003 & -0.2585 & -0.1003 \\ -0.1003 & 0.6327 & -0.1003 & -0.2585 \\ -0.2585 & -0.1003 & 0.6906 & -0.0713 \\ -0.1003 & -0.2585 & -0.0713 & 0.6906 \end{bmatrix}$$

Utilizing the direct assembly-superposition method with the element-to-global node assignment relations, the global conductance matrix is

$$[K] = \begin{bmatrix} 0.6906 & -0.1003 & 0 & -0.0713 & -0.2585 & 0 & 0 & 0 & 0 \\ -0.1003 & 1.2654 & -0.1003 & -0.2585 & -0.2006 & -0.2585 & 0 & 0 & 0 \\ 0 & -0.1003 & 0.6906 & 0 & -0.2585 & -0.0713 & 0 & 0 & 0 \\ -0.0713 & -0.2585 & 0 & 1.3812 & -0.2006 & 0 & -0.0713 & -0.2585 & 0 \\ -0.2585 & -0.2006 & -0.2585 & -0.2006 & 2.5308 & -0.2006 & -0.2585 & -0.2006 & -0.2585 \\ 0 & -0.2585 & -0.0713 & 0 & -0.2006 & 1.3812 & 0 & -0.2585 & -0.0713 \\ 0 & 0 & 0 & -0.0713 & -0.2585 & 0 & 0.7484 & -0.2585 & 0 \\ 0 & 0 & 0 & -0.2585 & -0.2006 & -0.2585 & -0.2585 & 1.3812 & -0.0713 \\ 0 & 0 & 0 & 0 & -0.2585 & -0.0713 & 0 & -0.0713 & 0.7484 \end{bmatrix}$$

The nodal temperature vector is

$$\{T\} = \begin{Bmatrix} 180 \\ 180 \\ 180 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{Bmatrix}$$

and we have explicitly incorporated the prescribed temperature boundary conditions.

Assembling the global force vector, noting that no internal heat is generated, we obtain

$$\{F\} = \begin{Bmatrix} 17.7084 + F_1 \\ 35.4168 + F_2 \\ 17.7084 + F_3 \\ 35.4168 \\ 47.2224 \\ 35.4168 \\ 23.6112 \\ 35.4168 \\ 23.6112 \end{Bmatrix} \text{Btu/hr}$$

where we use F_1, F_2, and F_3 as general notation to indicate that these are unknown "reaction" forces. In fact, as will be shown, these terms are the heat flux components at nodes 1, 2, and 3.

The global equations for the four-element model are then expressed as

$$\left[\begin{array}{ccc|cccccc}
0.6906 & -0.1003 & 0 & -0.0713 & -0.2585 & 0 & 0 & 0 & 0 \\
-0.1003 & 1.2654 & -0.1003 & -0.2585 & -0.2006 & -0.2585 & 0 & 0 & 0 \\
0 & -0.1003 & 0.6906 & 0 & -0.2585 & -0.0713 & 0 & 0 & 0 \\
\hline
-0.0713 & -0.2585 & 0 & 1.3812 & -0.2006 & 0 & -0.0713 & -0.2585 & 0 \\
-0.2585 & -0.2006 & -0.2585 & -0.2006 & 2.5308 & -0.2006 & -0.2585 & -0.2006 & -0.2585 \\
0 & -0.2585 & -0.0713 & 0 & -0.2006 & 1.3812 & 0 & -0.2585 & -0.0713 \\
0 & 0 & 0 & -0.0713 & -0.2585 & 0 & 0.7484 & -0.2585 & 0 \\
0 & 0 & 0 & -0.2585 & -0.2006 & -0.2585 & -0.2585 & 1.3812 & -0.0713 \\
0 & 0 & 0 & 0 & -0.2585 & -0.0713 & 0 & -0.0713 & 0.7484
\end{array}\right]
\left\{\begin{array}{c} 180 \\ 180 \\ 180 \\ \hline T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{array}\right\}
= \left\{\begin{array}{c} 17.7084 + F_1 \\ 35.4168 + F_2 \\ 17.7084 + F_3 \\ \hline 35.4168 \\ 47.2224 \\ 35.4168 \\ 23.6112 \\ 35.4158 \\ 23.6112 \end{array}\right\}$$

Taking into account the specified temperatures on nodes 1, 2, and 3, the global equations for the unknown temperatures become

$$\begin{bmatrix}
1.3812 & -0.2006 & 0 & -0.0713 & -0.2585 & 0 \\
-0.2006 & 2.5308 & -0.2006 & -0.2585 & -0.2006 & -0.2585 \\
0 & -0.2006 & 1.3812 & 0 & -0.2585 & -0.0713 \\
-0.0713 & -0.2585 & 0 & 0.7484 & -0.2585 & 0 \\
-0.2585 & -0.2006 & -0.2585 & -0.2585 & 1.3812 & -0.0713 \\
0 & -0.2585 & -0.0713 & 0 & -0.0713 & 0.7484
\end{bmatrix}
\begin{Bmatrix} T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{Bmatrix}
= \begin{Bmatrix} 94.7808 \\ 176.3904 \\ 94.7808 \\ 23.6112 \\ 35.4168 \\ 23.6112 \end{Bmatrix}$$

The reader is urged to note that, in arriving at the last result, we partition the global matrix as shown by the dashed lines and apply Equation 3.46a to obtain the equations governing the "active" degrees of freedom. That is, the partitioned matrix is of the form

$$\begin{bmatrix} K_{cc} & K_{ca} \\ K_{ac} & K_{aa} \end{bmatrix} \begin{Bmatrix} T_c \\ T_a \end{Bmatrix} = \begin{Bmatrix} F_c \\ F_a \end{Bmatrix}$$

where the subscript c denotes terms associated with constrained (specified) temperatures and the subscript a denotes terms associated with active (unknown) temperatures. Hence, this 6×6 system represents

$$[K_{aa}]\{T_a\} = \{F_a\} - [K_{ac}]\{T_c\}$$

which now properly includes the effects of specified temperatures as forcing functions on the right-hand side.

Simultaneous solution of the global equations (in this case, we inverted the global stiffness matrix using a spreadsheet program) yields the nodal temperatures as

$$\begin{Bmatrix} T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{Bmatrix} = \begin{Bmatrix} 106.507 \\ 111.982 \\ 106.507 \\ 89.041 \\ 90.966 \\ 89.041 \end{Bmatrix} \ ^\circ\text{F}$$

If we now back substitute the computed nodal temperatures into the first three of the global equations, specifically,

$$0.6906T_1 - 0.1003T_2 - 0.0713T_4 - 0.2585T_5 = 17.7084 + F_1$$
$$-0.1003T_1 + 1.2654T_2 - 0.1003T_3 - 0.2585T_4 - 0.2006T_5 - 0.2585T_6 = 35.4168 + F_2$$
$$-0.1003T_2 + 0.6906T_3 - 0.2585T_5 - 0.0713T_6 = 17.7084 + F_3$$

we obtain the heat flow values at nodes 1, 2, and 3 as

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 52.008 \\ 78.720 \\ 52.008 \end{Bmatrix} \text{Btu/hr}$$

Note that, in terms of the matrix partitioning, we are now solving

$$[K_{cc}]\{T_c\} + [K_{ca}]\{T_a\} = \{F_c\}$$

to obtain the unknown values in $\{F_c\}$.

Since there is no convection from the edges defined by nodes 1-2 and 1-3 and the temperature is specified on these edges, the reaction "forces" represent the heat input (flux) across these edges and should be in balance with the convection loss across the lateral surfaces of the body, and its edges, in a steady-state situation. This balance is a check that can and should be made on the accuracy of a finite element solution of a heat transfer problem and is analogous to checking equilibrium of a structural finite element solution.

Example 7.5 is illustrated in great detail to point out the systematic procedures for assembling the global matrices and force vectors. The astute reader ascertains, in following the solution, that symmetry conditions can be used to simplify the mathematics of the solution. As shown in Figure 7.11a, an axis (plane) of symmetry exists through the horizontal center of the plate. Therefore, the problem can be reduced to a two-element model, as shown in Figure 7.11b. Along the edge of symmetry, the y-direction heat flux components are in balance, and this edge can be treated as a perfectly insulated edge. One could then use only two elements, with the appropriately adjusted boundary conditions to obtain the same solution as in the example.

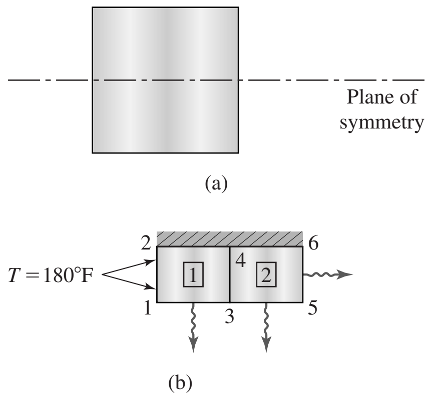

Figure 7.11 Model of Example 7.5, showing (a) the plane of symmetry and (b) a two-element model with adjusted boundary conditions.

7.4.3 Symmetry Conditions

As mentioned previously in connection with Example 7.5, symmetry conditions can be used to reduce the size of a finite element model (or any other computational model). Generally, the symmetry is observed geometrically; that is, the physical domain of interest is symmetric about an axis or plane. Geometric symmetry is not, however, sufficient to ensure that a problem is symmetric. In addition, the boundary conditions and applied loads must be symmetric about the axis or plane of geometric symmetry as well. To illustrate, consider Figure 7.12a, depicting a thin rectangular plate having a heat source located at the geometric center of the plate. The model is of a heat transfer fin removing heat from a central source (a pipe containing hot fluid, for example) via conduction and convection from the fin. Clearly, the situation depicted is symmetric geometrically. But, is the situation a symmetric problem? The loading is symmetric, since the heat source is centrally located in the domain. We also assume that $k_x = k_y$ so that the material properties are symmetric. Hence, we must examine the boundary conditions to determine if symmetry exists. If, for example, as shown in Figure 7.12b, the ambient temperatures external to the fin are uniform around the fin and the convection coefficients are the same on all surfaces, the problem is symmetric about both x and y axes and can be solved via the model in Figure 7.12c. For this situation, note that the heat from the source is conducted radially and, consequently, across the x axis, the heat flux q_y is zero and, across the y axis, the heat flux q_x must also be zero. These observations reveal the boundary conditions for the quarter-symmetry model shown in Figure 7.12d and the internal forcing function is taken as $Q/4$. On the other hand, let us assume that the upper edge of the plate is perfectly insulated, as in Figure 7.12e. In this case, we do not have

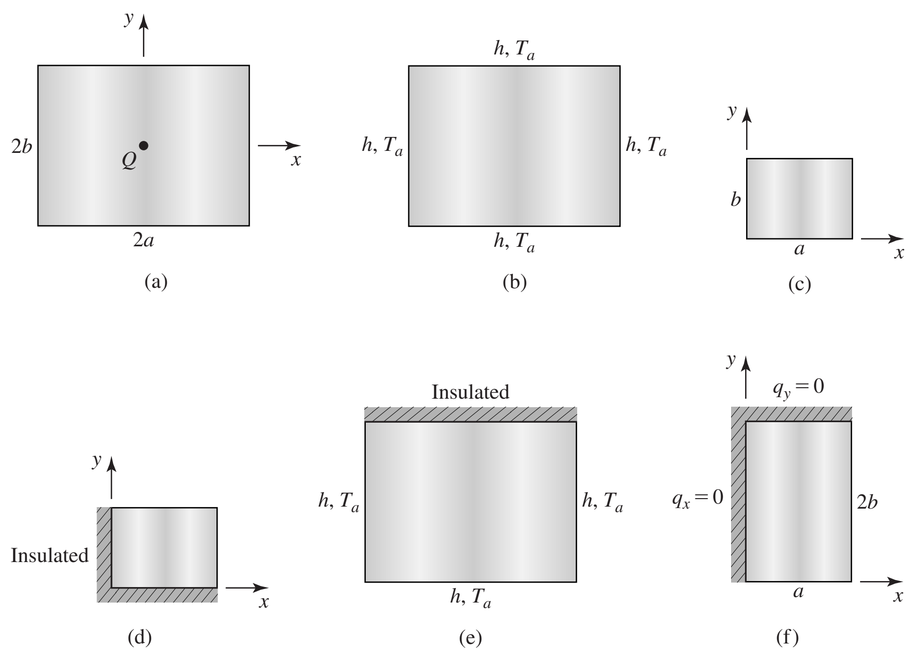

Figure 7.12 Illustrations of symmetry dictated by boundary conditions.

symmetric conditions about the x axis but symmetry about the y axis exists. For these conditions, we can use the "half-symmetry" model shown in Figure 7.12f, using the symmetry (boundary) condition $q_x = 0$ across $x = 0$ and apply the internal heat generation term $Q/2$.

Symmetry can be used to reduce the size of finite element models significantly. It must be remembered that symmetry is not simply a geometric occurrence. For symmetry, geometry, loading, material properties, and boundary conditions must all be symmetric (about an axis, axes, or plane) to reduce the model.

7.4.4 Element Resultants

In the approach just taken in heat transfer analysis, the primary nodal variable computed is temperature. Most often in such analyses, we are more interested in the amount of heat transferred than the nodal temperatures. (This is analogous to structural problems: We solve for nodal displacements but are more interested in stresses.) In finite element analyses of heat transfer problems, we must back substitute the nodal temperature solution into the "reaction" equations to obtain global heat transfer values. (As in Example 7.5, when we solved the partitioned matrices for the heat flux values at the constrained nodes.) Similarly, we can back substitute the nodal temperatures to obtain estimates of heat transfer properties of individual elements as well.

The heat flux components for a two-dimensional element, per Fourier's law, are

$$
\begin{aligned}
q_x^{(e)} &= -k_x \frac{\partial T^{(e)}}{\partial x} = -k_x \sum_{i=1}^{M} \frac{\partial N_i}{\partial x} T_i^{(e)} \\
q_y^{(e)} &= -k_y \frac{\partial T^{(e)}}{\partial y} = -k_y \sum_{i=1}^{M} \frac{\partial N_i}{\partial y} T_i^{(e)}
\end{aligned}
\tag{7.47}
$$

where we again denote the total number of element nodes as M. With the exception of the three-node triangular element, the flux components given by Equation 7.47 are not constant but vary with position in the element. As an example, the components for the four-node rectangular element are readily computed using the interpolation functions of Equation 6.56, repeated here as

$$
\begin{aligned}
N_1(r, s) &= \frac{1}{4}(1 - r)(1 - s) \\
N_2(r, s) &= \frac{1}{4}(1 + r)(1 - s) \\
N_3(r, s) &= \frac{1}{4}(1 + r)(1 + s) \\
N_4(r, s) &= \frac{1}{4}(1 - r)(1 + s)
\end{aligned}
\tag{7.48}
$$

Recalling that

$$
\frac{\partial}{\partial x} = \frac{1}{a}\frac{\partial}{\partial r} \quad \text{and} \quad \frac{\partial}{\partial y} = \frac{1}{b}\frac{\partial}{\partial s}
$$

we have

$$
\begin{aligned}
q_x^{(e)} &= -\frac{k_x}{a} \sum_{i=1}^{4} \frac{\partial N_i}{\partial r} T_i^{(e)} \\
&= -\frac{k_x}{4a}\left[(s - 1)T_1^{(e)} + (1 - s)T_2^{(e)} + (1 + s)T_3^{(e)} - (1 + s)T_4^{(e)}\right] \\
q_y^{(e)} &= -\frac{k_y}{b} \sum_{i=1}^{4} \frac{\partial N_i}{\partial s} T_i^{(e)} \\
&= -\frac{k_y}{4b}\left[(r - 1)T_1^{(e)} - (1 + r)T_2^{(e)} + (1 + r)T_3^{(e)} + (1 - r)T_4^{(e)}\right]
\end{aligned}
\tag{7.49}
$$

and these expressions simplify to

$$q_x^{(e)} = -\frac{k_x}{4a}\left[(1-s)\left(T_2^{(e)} - T_1^{(e)}\right) + (1+s)\left(T_3^{(e)} - T_4^{(e)}\right)\right]$$
$$q_y^{(e)} = -\frac{k_y}{4b}\left[(1-r)\left(T_4^{(e)} - T_1^{(e)}\right) + (1+r)\left(T_3^{(e)} - T_2^{(e)}\right)\right] \tag{7.50}$$

The flux components, therefore the temperature gradients, vary linearly in a four-node rectangular element. However, recall that, for a C^0 formulation, the gradients are not, in general, continuous across element boundaries. Consequently, the element flux components associated with an individual element are customarily taken to be the values calculated at the centroid of the element. For the rectangular element, the centroid is located at $(r, s) = (0, 0)$, so the centroidal values are simply

$$q_x^{(e)} = -\frac{k_x}{4a}\left(T_2^{(e)} + T_3^{(e)} - T_1^{(e)} - T_4^{(e)}\right)$$
$$q_y^{(e)} = -\frac{k_y}{4b}\left(T_3^{(e)} + T_4^{(e)} - T_1^{(e)} - T_2^{(e)}\right) \tag{7.51}$$

The centroidal values calculated per Equation 7.51, in general, are quite accurate for a fine mesh of elements. Some finite element software packages compute the values at the integration points (the Gauss points) and average those values for an element value to be applied at the element centroid. In either case, the computed values are needed to determine solution convergence and should be checked at every stage of a finite element analysis.

EXAMPLE 7.6

Calculate the centroidal heat flux components for elements 2 and 3 of Example 7.5.

■ **Solution**

From Example 7.4, we have $a = b = 0.5$ in., $k_x = k_y = 20$ Btu/(hr-ft-°F), and from Example 7.5, the nodal temperature vector is

$$\{T\} = \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{Bmatrix} = \begin{Bmatrix} 180 \\ 180 \\ 180 \\ 106.507 \\ 111.982 \\ 106.507 \\ 89.041 \\ 90.966 \\ 89.041 \end{Bmatrix} \text{°F}$$

For element 2, the element-global nodal correspondence relation can be written as

$$\left[T_1^{(2)} \quad T_2^{(2)} \quad T_3^{(2)} \quad T_4^{(2)}\right] = [T_4 \quad T_7 \quad T_8 \quad T_5]$$
$$= [106.507 \quad 89.041 \quad 90.966 \quad 111.982]$$

Substituting numerical values into Equation 7.49,

$$q_x^{(2)} = -\frac{12(20)}{4(0.5)}(89.041 + 90.966 - 106.507 - 111.982) = 4617.84\ \text{Btu/(hr-ft}^2)$$

$$q_y^{(2)} = -\frac{12(20)}{4(0.5)}(90.966 + 111.982 - 106.507 - 89.041) = -888.00\ \text{Btu/(hr-ft}^2)$$

and, owing to the symmetry conditions, we have

$$q_x^{(3)} = 4617.84\ \text{Btu/(hr-ft}^2)$$
$$q_y^{(3)} = 888.00\ \text{Btu/(hr-ft}^2)$$

as may be verified by direct calculation. Recall that these values are calculated at the location of the element centroid.

The element resultants representing convection effects can also be readily computed once the nodal temperature solution is known. The convection resultants are of particular interest, since these represent the primary source of heat removal (or absorption) from a solid body. The convective heat flux, per Equation 7.2, is

$$q_x = h(T - T_a)\ \text{Btu/(hr-ft}^2)\ \text{or W/m}^2 \tag{7.52}$$

where all terms are as previously defined. Hence, the total convective heat flow rate from a surface area A is

$$\dot{H}_h = \iint\limits_A h(T - T_a)\,\mathrm{d}A \tag{7.53}$$

For an individual element, the heat flow rate is

$$\dot{H}_h^{(e)} = \iint\limits_A h(T^{(e)} - T_a)\,\mathrm{d}A = \iint\limits_A h([N]\{T\} - T_a)\,\mathrm{d}A \tag{7.54}$$

The area of integration in Equation 7.54 includes all portions of the element surface subjected to convection conditions. In the case of a two-dimensional element, the area may include lateral surfaces (that is, convection perpendicular to the plane of the element) as well as the area of element edges located on a free boundary.

EXAMPLE 7.7

Determine the total heat flow rate of convection for element 3 of Example 7.5.

■ Solution

First we note that, for element 3, the element-to-global correspondence relation for nodal temperatures is

$$\left[T_1^{(3)} \quad T_2^{(3)} \quad T_3^{(3)} \quad T_4^{(3)}\right] = [T_5 \quad T_8 \quad T_9 \quad T_6]$$

Second, element 3 is subjected to convection on both lateral surfaces as well as the two edges defined by nodes 8-9 and 6-9. Consequently, three integrations are required as follows:

$$\dot{H}_h^{(e)} = 2\iint\limits_{A^{(e)}} h([N]\{T\} - T_a)\,\mathrm{d}A^{(e)} + \iint\limits_{A_{8-9}} h([N]\{T\} - T_a)\,\mathrm{d}A_{8-9} + \iint\limits_{A_{6-9}} h([N]\{T\} - T_a)\,\mathrm{d}A_{6-9}$$

where $A^{(e)}$ is element area in the xy plane and the multiplier in the first term (2) accounts for both lateral surfaces.

Transforming the first integral to normalized coordinates results in

$$I_1 = 2hab\int_{-1}^{1}\int_{-1}^{1}([N]\{T\} - T_a)\,\mathrm{d}r\,\mathrm{d}s = 2hab\int_{-1}^{1}\int_{-1}^{1}[N]\,\mathrm{d}r\,\mathrm{d}s\,\{T\} - 2habT_a\int_{-1}^{1}\mathrm{d}r\,\mathrm{d}s$$

$$= \frac{2hA}{4}\int_{-1}^{1}\int_{-1}^{1}[N]\,\mathrm{d}r\,\mathrm{d}s\,\{T\} - 2hAT_a$$

Therefore, we need integrate the interpolation functions only over the area of the element, as all other terms are known constants. For example,

$$\int_{-1}^{1}\int_{-1}^{1} N_1\,\mathrm{d}r\,\mathrm{d}s = \int_{-1}^{1}\int_{-1}^{1}\frac{1}{4}(1-r)(1-s)\,\mathrm{d}r\,\mathrm{d}s = \frac{1}{4}\frac{(1-r)^2}{2}\bigg|_{-1}^{1}\frac{(1-s)^2}{2}\bigg|_{-1}^{1} = 1$$

An identical result is obtained when the other three functions are integrated. The integral corresponding to convection from the element lateral surfaces is then

$$I_1 = 2hA\left(\frac{T_1^{(3)} + T_2^{(3)} + T_3^{(3)} + T_4^{(3)}}{4} - T_a\right)$$

The first term in the parentheses is the average of the nodal temperatures, and this is a general result for the rectangular element. Substituting numerical values

$$I_1 = \frac{2(50)(1)^2}{144}\left(\frac{111.982 + 90.966 + 89.041 + 106.507}{4} - 68\right) = 21.96 \text{ Btu/hr}$$

Next, we consider the edge convection terms. Along edge 8-9,

$$I_2 = \iint\limits_{A_{8-9}} h([N]\{T\} - T_a)\,\mathrm{d}A_{8-9}$$

and, since $r = 1$ along that edge, $\mathrm{d}A_{8-9} = tb\,\mathrm{d}s$, and the integral becomes

$$I_2 = htb\int_{-1}^{1}([N_{r=1}]\{T\} - T_a\,\mathrm{d}s)$$

$$= htb\int_{-1}^{1}\frac{1}{4}[0 \quad 1-s \quad 1+s \quad 0]\,\mathrm{d}s\,\{T\} - htbT_a\int_{-1}^{1}\mathrm{d}s$$

$$= h(2tb)\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}\{T\} - h(2tb)T_a$$

$$= hA_{\text{edge}}\left(\frac{T_2^{(3)} + T_3^{(3)}}{2} - T_a\right)$$

Again, we observe that the average temperature of the nodes associated with the area of the edge appears. Stated another way, the convection area is allocated equally to the two nodes, and this is another general result for the rectangular element. Inserting numerical values,

$$I_2 = \frac{50(0.5)(1)}{144}\left(\frac{90.966 + 89.041}{2} - 68\right) = 3.82 \text{ Btu/hr}$$

By analogy, the edge convection along edge 6-9 is

$$I_3 = hA_{\text{edge}}\left(\frac{T_3^{(3)} + T_4^{(3)}}{2} - T_a\right) = \frac{50(0.5)(1)}{144}\left(\frac{89.041 + 106.507}{2} - 68\right)$$

$$= 5.17 \text{ Btu/hr}$$

The total convective heat flow rate for element 3 is then

$$\dot{H}_h^{(3)} = I_1 + I_2 + I_3 = 30.95 \text{ Btu/hr}$$

7.4.5 Internal Heat Generation

To this point in the current discussion of heat transfer, only examples having no internal heat generation ($Q = 0$) have been considered. Also, for two-dimensional heat transfer, we considered only thin bodies such as fins. Certainly these are not the only cases of interest. Consider the situation of a body of constant cross section having length much larger than the cross-sectional dimensions, as shown in Figure 7.13a (we use a rectangular cross section for convenience). In addition, an internal heat source is imbedded in the body and runs parallel to the length. Practical examples include a floor slab containing a hot water or steam pipe for heating and a sidewalk or bridge deck having embedded heating cables to prevent ice accumulation. The internal heat generation source in this situation is known as a *line* source.

Except very near the ends of such a body, heat transfer effects in the z direction can be neglected and the situation treated as a two-dimensional problem, as depicted in Figure 7.13b. Assuming the pipe or heat cable to be small in comparison to the cross section of the body, the source is treated as acting at a single point in the cross section. If we model the problem via the finite element method, how do we account for the source in the formulation? Per the first of Equation 7.38, the nodal force vector corresponding to internal heat generation is

$$\left\{f_Q^{(e)}\right\} = \iint_A Q[N]^T t \, \mathrm{d}A = \iint_A Q\{N\} t \, \mathrm{d}A \tag{7.55}$$

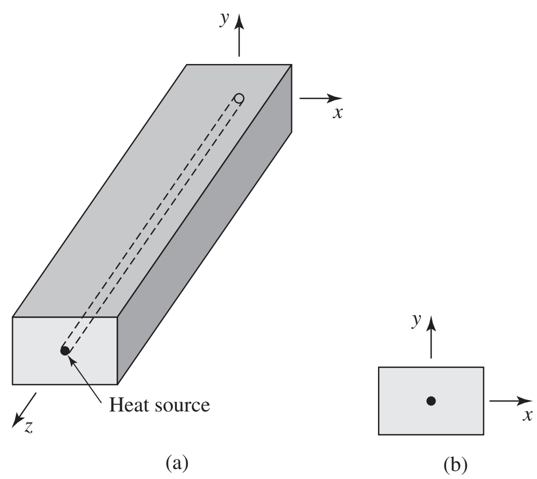

Figure 7.13
(a) Long, slender body with internal heat source. (b) 2-D representation (unit thickness in *z*-direction).

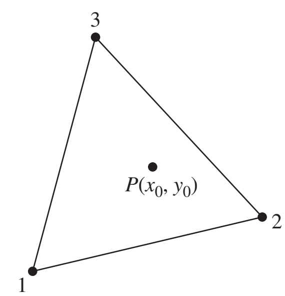

Figure 7.14
Concentrated heat source Q^* located at point $P(x_0, y_0)$ in a triangular element.

where, as before, t is element thickness. In this type of problem, it is customary to take t as unity, so that all computations are *per unit length*. In accordance with this convention, the source strength is denoted Q^*, having units typically expressed as Btu/(hr-ft^2) or W/m^2. Equation 7.55 then becomes

$$\left\{ f_Q^{(e)} \right\} = \iint_A Q^*[N]^T \, \mathrm{d}A = \iint_A Q^*\{N\} \, \mathrm{d}A \tag{7.56}$$

The question is now mathematical: How do we integrate a function applicable at a single point in a two-dimensional domain? Mathematically, the operation is quite simple if the concept of the *Dirac delta* or *unit impulse* function is introduced. We choose not to take the strictly mathematical approach, however, in the interest of using an approach based on logic and all the foregoing information presented on interpolation functions.

For illustrative purposes, the heat source is assumed to be located at a known point $P = (x_0, y_0)$ in the interior of a three-node triangular element, as in Figure 7.14. If we know the temperature at each of the three nodes of the element, then the temperature at point P is a weighted combination of the nodal temperatures. By this point in the text, the reader is well aware that the weighting factors are the interpolation functions. If nodal values are interpolated to a specific point, a value at that point should properly be assigned to the nodes via the same interpolation functions evaluated at the point. Using this premise, the nodal forces for the triangular element become (assuming Q^* to be constant)

$$\left\{f_Q^{(e)}\right\} = Q^* \iint_A \begin{Bmatrix} N_1(x_0, y_0) \\ N_2(x_0, y_0) \\ N_3(x_0, y_0) \end{Bmatrix} \mathrm{d}A \tag{7.57}$$

For a three-node triangular element, the interpolation functions (from Chapter 6) are simply the area coordinates, so we now have

$$\left\{f_Q^{(e)}\right\} = Q^* \iint_A \begin{Bmatrix} L_1(x_0, y_0) \\ L_2(x_0, y_0) \\ L_3(x_0, y_0) \end{Bmatrix} \mathrm{d}A = Q^* A \begin{Bmatrix} L_1(x_0, y_0) \\ L_2(x_0, y_0) \\ L_3(x_0, y_0) \end{Bmatrix} \tag{7.58}$$

Now consider the "behavior" of the area coordinates as the position of the interior point P varies in the element. As P approaches node 1, for example, area coordinate L_1 approaches unity value. Clearly, if the source is located at node 1, the entire source value should be allocated to that node. A similar argument can be made for each of the other nodes. Another very important point to observe here is that the total heat generation as allocated to the nodes by Equation 7.58 is equivalent to the source. If we sum the individual nodal contributions given in Equation 7.58, we obtain

$$\sum_{i=1}^{3} Q_i^{*(e)} = \sum_{i=1}^{3} \left(L_1^{(e)} + L_2^{(e)} + L_3^{(e)}\right) Q^* A = Q^* A \tag{7.59}$$

since $\sum_{i=1}^{3} L_i = 1$ is known by the definition of area coordinates.

The foregoing approach using logic and our knowledge of interpolation functions is without mathematical rigor. If we approach the situation of a line source mathematically, the result is exactly the same as that given by Equation 7.58 for the triangular element. For any element chosen, the force vector corresponding to a line source (keep in mind that, in two-dimensions, this looks like a point source) the nodal force contributions are

$$\left\{f_Q^{(e)}\right\} = Q^* \iint_A \{N(x_0, y_0)\} \, \mathrm{d}A \tag{7.60}$$

Thus, a source of internal heat generation is readily allocated to the nodes of a finite element via the interpolation functions of the specific element applied.

7.5 HEAT TRANSFER WITH MASS TRANSPORT

The finite element formulations and examples previously presented deal with solid media in which heat flows as a result of conduction and convection. An additional complication arises when the medium of interest is a flowing fluid. In such a case, heat flows by conduction, convection, and via motion of the media. The last effect, referred to as *mass transport,* is considered here for the

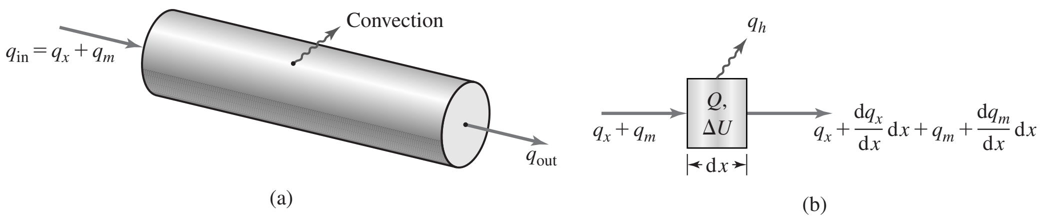

Figure 7.15
(a) One-dimensional conduction with convection and mass transport. (b) Control volume for energy balance.

one-dimensional case. Figure 7.15a is essentially Figure 7.2a with a major physical difference. The volume shown in Figure 7.15a represents a flowing fluid (as in a pipe, for example) and heat is transported as a result of the flow. The heat flux associated with mass transport is denoted q_m, as indicated in the figure. The additional flux term arising from mass transport is given by

$$q_m = \dot{m}cT \text{ (W or Btu/hr)} \tag{7.61}$$

where $\dot{m}$ is mass flow rate (kg/hr or slug/hr), c is the specific heat of the fluid (W-hr/(kg-°C) or Btu/(slug-°F)), and $T(x)$ is the temperature of the fluid (°C or °F). A control volume of length dx of the flow is shown in Figure 7.15b, where the flux terms have been expressed as two-term Taylor series as in past derivations. Applying the principle of conservation of energy (in analogy with Equation 7.1),

$$q_x A\,dt + q_m\,dt + QA\,dx\,dt = \Delta U + \left(q_x + \frac{dq_x}{dx}\,dx\right)A\,dt + \left(q_m + \frac{dq_m}{dx}\,dx\right)dt + q_h P\,dx\,dt \tag{7.62}$$

Considering steady-state conditions, $\Delta U = 0$, using Equations 5.51 and 7.2 and simplifying yields

$$\frac{d}{dx}\left(k_x\frac{dT}{dx}\right) + Q = \frac{dq_m}{dx} + \frac{hP}{A}(T - T_a) \tag{7.63}$$

where all terms are as previously defined. Substituting for q_m into Equation 7.63, we obtain

$$\frac{d}{dx}\left(k_x\frac{dT}{dx}\right) + Q = \frac{d}{dx}\left(\frac{\dot{m}c}{A}T\right) + \frac{hP}{A}(T - T_a) \tag{7.64}$$

which for constant material properties and constant mass flow rate (steady state) becomes

$$k_x\frac{d^2T}{dx^2} + Q = \frac{\dot{m}c}{A}\frac{dT}{dx} + \frac{hP}{A}(T - T_a) \tag{7.65}$$

With the exception of the mass transport term, Equation 7.65 is identical to Equation 7.4. Consequently, if we apply Galerkin's finite element method, the procedure and results are identical to those of Section 7.3, except for additional stiffness matrix terms arising from mass transport. Rather than repeat the derivation of known terms, we develop only the additional terms. If Equation 7.65 is substituted into the residual equations for a two-node linear element (Equation 7.6), the additional terms are

$$\int_{x_1}^{x_2} \dot{m}c \frac{\mathrm{d}T}{\mathrm{d}x} N_i \, \mathrm{d}x \qquad i = 1, 2 \tag{7.66}$$

Substituting for T via Equation 7.5, this becomes

$$\int_{x_1}^{x_2} \dot{m}c \left[\frac{\mathrm{d}N_1}{\mathrm{d}x} T_1 + \frac{\mathrm{d}N_2}{\mathrm{d}x} T_2 \right] N_i \, \mathrm{d}x \qquad i = 1, 2 \tag{7.67}$$

Therefore, the additional stiffness matrix resulting from mass transport is

$$[k_{\dot{m}}] = \dot{m}c \int_{x_1}^{x_2} \begin{bmatrix} N_1 \dfrac{\mathrm{d}N_1}{\mathrm{d}x} & N_1 \dfrac{\mathrm{d}N_2}{\mathrm{d}x} \\ N_2 \dfrac{\mathrm{d}N_1}{\mathrm{d}x} & N_2 \dfrac{\mathrm{d}N_2}{\mathrm{d}x} \end{bmatrix} \mathrm{d}x \tag{7.68}$$

EXAMPLE 7.8

Explicitly evaluate the stiffness matrix given by Equation 7.68 for the two-node element.

■ **Solution**

The interpolation functions are

$$N_1 = 1 - \frac{x}{L}$$
$$N_2 = \frac{x}{L}$$

and the required derivatives are

$$\frac{\mathrm{d}N_1}{\mathrm{d}x} = -\frac{1}{L}$$
$$\frac{\mathrm{d}N_2}{\mathrm{d}x} = \frac{1}{L}$$

Utilizing the change of variable $s = x/L$, Equation 7.68 becomes

$$[k_{\dot{m}}] = \frac{\dot{m}c}{L} \int_0^1 \begin{bmatrix} -(1-s) & (1-s) \\ -s & s \end{bmatrix} L \, \mathrm{d}s = \dot{m}c \begin{bmatrix} -\dfrac{1}{2} & \dfrac{1}{2} \\ -\dfrac{1}{2} & \dfrac{1}{2} \end{bmatrix} = \frac{\dot{m}c}{2} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

and note that the matrix is not symmetric.

Using the result of Example 7.8, the stiffness matrix for a one-dimensional heat transfer element with conduction, convection, and mass transport is given by

$$\left[k^{(e)}\right] = \frac{k_x A}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{hPL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \frac{\dot{m}c}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \left[k_c^{(e)}\right] + \left[k_h^{(e)}\right] + \left[k_{\dot{m}}^{(e)}\right] \tag{7.69}$$

where the conduction and convection terms are identical to those given in Equation 7.15. Note that the forcing functions and boundary conditions for the one-dimensional problem with mass transport are the same as given in Section 7.3, Equations 7.16 through 7.19.

EXAMPLE 7.9

Figure 7.16a shows a thin-walled tube that is part of an oil cooler. Engine oil enters the tube at the left end at temperature 50°C with a flow rate of 0.2 kg/min. The tube is surrounded by air flowing at a constant temperature of 15°C. The thermal properties of the oil are as follows:

Thermal conductivity: $k_x = 0.156$ W/(m-°C)

Specific heat: $c = 0.523$ W-hr/(kg-°C)

The convection coefficient between the thin wall and the flowing air is $h = 300$ W/(m^2-°C). The tube wall thickness is such that conduction effects in the wall are to be neglected; that is, the wall temperature is constant through its thickness and the same as the temperature of the oil in contact with the wall at any position along the length of the tube. Using four two-node finite elements, obtain an approximate solution for the temperature distribution along the length of the tube and determine the heat removal rate via convection.

■ Solution

The finite element model is shown schematically in Figure 7.16b, using equal length elements $L = 25$ cm $= 0.025$ m. The cross-sectional area is $A = (\pi/4)(20/1000)^2 =$

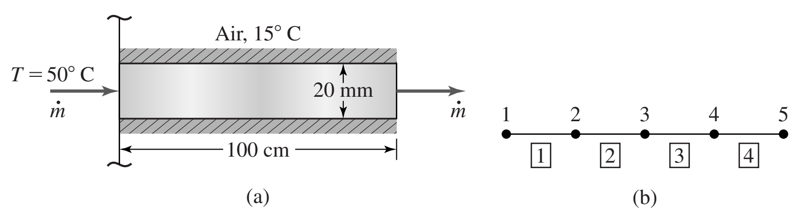

Figure 7.16
(a) Oil cooler tube of Example 7.9. (b) Element and node numbers for a four-element model.

$3.14(10^{-4})$ m^2. And the peripheral dimension (circumference) of each element is $P = \pi(20/1000) = 6.28(10^{-2})$ m. The stiffness matrix for each element (note that all elements are identical) is computed via Equation 7.69 as follows:

$$\left[k_c^{(e)}\right] = \frac{k_x A}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{0.156(3.14)(10^{-4})}{0.025}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.9594 & -1.9594 \\ -1.9594 & 1.9594 \end{bmatrix}(10^{-3})$$

$$\left[k_h^{(e)}\right] = \frac{hPL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{300(6.28)(10^{-2})(0.025)}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0.157 & 0.0785 \\ 0.0785 & 0.157 \end{bmatrix}$$

$$\left[k_{\dot{m}}^{(e)}\right] = \frac{\dot{m}c}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{(0.2)(60)(0.523)}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -3.138 & 3.138 \\ -3.138 & 3.138 \end{bmatrix}$$

$$\left[k^{(e)}\right] = \begin{bmatrix} -2.9810 & 3.2165 \\ -3.0595 & 3.2950 \end{bmatrix}$$

At this point, note that the mass transport effects dominate the stiffness matrix and we anticipate that very little heat is dissipated, as most of the heat is carried away with the flow. Also observe that, owing to the relative magnitudes, the conduction effects have been neglected.

Assembling the global stiffness matrix via the now familiar procedure, we obtain

$$[K] = \begin{bmatrix} -2.9810 & 3.2165 & 0 & 0 & 0 \\ -3.0595 & 0.314 & 3.2165 & 0 & 0 \\ 0 & -3.0595 & 0.314 & 3.2165 & 0 \\ 0 & 0 & -3.0595 & 0.314 & 3.2165 \\ 0 & 0 & 0 & -3.0595 & 3.2950 \end{bmatrix}$$

The convection-driven forcing function for each element per Equation 7.18 is

$$\left\{f_h^{(e)}\right\} = \frac{hPT_aL}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \frac{300(6.28)(10^{-2})(15)(0.025)}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 3.5325 \\ 3.5325 \end{Bmatrix}$$

As there is no internal heat generation, the per-element contribution of Equation 7.16 is zero. Finally, we must examine the boundary conditions. At node 1, the temperature is specified but the heat flux $q_1 = F_1$ is unknown; at node 5 (the exit), the flux is also unknown. Unlike previous examples, where a convection boundary condition existed, here we assume that the heat removed at node 5 is strictly a result of mass transport. Physically, this means we define the problem such that heat transfer ends at node 5 and the heat remaining in the flow at this node (the exit) is carried away to some other process. Consequently, we do not consider either a conduction or convection boundary condition at node 5. Instead, we compute the temperature at node 5 then the heat removed at this node via the mass transport relation. In terms of the finite element model, this means that we do not consider the heat flow through node 5 as an unknown (reaction force). With this in mind, we assemble the global force vector from the element force vectors to obtain

$$\{F\} = \begin{Bmatrix} 3.5325 + F_1 \\ 7.065 \\ 7.065 \\ 7.065 \\ 3.5325 \end{Bmatrix}$$

The assembled system (global) equations are then

$$[K]\{T\} = \begin{bmatrix} -2.9801 & 3.2165 & 0 & 0 & 0 \\ -3.0595 & 0.314 & 3.2165 & 0 & 0 \\ 0 & -3.0595 & 0.314 & 3.2165 & 0 \\ 0 & 0 & -3.0595 & 0.314 & 3.2165 \\ 0 & 0 & 0 & -3.0595 & 3.2950 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix}$$

$$= \begin{Bmatrix} 3.5325 + F_1 \\ 7.065 \\ 7.065 \\ 7.065 \\ 3.5325 \end{Bmatrix}$$

Applying the known condition at node 1, $T = 50°\text{C}$, the reduced system equations become

$$\begin{bmatrix} 0.314 & 3.2165 & 0 & 0 \\ -3.0595 & 0.314 & 3.2165 & 0 \\ 0 & -3.0595 & 0.314 & 3.2165 \\ 0 & 0 & -3.0595 & 3.2950 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 160.04 \\ 7.065 \\ 7.065 \\ 3.5325 \end{Bmatrix}$$

which yields the solution for the nodal temperatures as

$$\begin{Bmatrix} T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 47.448 \\ 45.124 \\ 42.923 \\ 40.928 \end{Bmatrix} °\text{C}$$

As conduction effects have been seen to be negligible, the input rate is computed as

$$q_{\text{in}} = q_{m1} = \dot{m}cT_1 = 0.2(60)(0.523)(50) = 3138 \text{ W}$$

while, at node 5, the output rate is

$$q_{m5} = \dot{m}cT_5 = 0.2(60)(0.523)(40.928) = 2568.6 \text{ W}$$

The results show that only about 18 percent of input heat is removed, so the cooler is not very efficient.

7.6 HEAT TRANSFER IN THREE DIMENSIONS

As the procedure has been established, the governing equation for heat transfer in three dimensions is not derived in detail here. Instead, we simply present the equation as

$$\frac{\partial}{\partial x}\left(k_x\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_z\frac{\partial T}{\partial z}\right)+Q=0 \tag{7.70}$$

and note that only conduction effects are included and steady-state conditions are assumed. In the three-dimensional case, convection effects are treated most efficiently as boundary conditions, as is discussed.

The domain to which Equation 7.70 applies is represented by a mesh of finite elements in which the temperature distribution is discretized as

$$T(x, y, z)=\sum_{i=1}^{M}N_i(x, y, z)T_i=[N]\{T\} \tag{7.71}$$

where M is the number of nodes per element. Application of the Galerkin method to Equation 7.70 results in M residual equations:

$$\iiint_V\left[\frac{\partial}{\partial x}\left(k_x\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_z\frac{\partial T}{\partial z}\right)+Q\right]N_i\,\mathrm{d}V=0$$

$$i=1,\ldots,M \tag{7.72}$$

where, as usual, V is element volume.

In a manner analogous to Section 7.4 for the two-dimensional case, the derivative terms can be written as

$$\begin{aligned}
\frac{\partial}{\partial x}\left(k_x\frac{\partial T}{\partial x}\right)N_i&=\frac{\partial}{\partial x}\left(k_x\frac{\partial T}{\partial x}N_i\right)-k_x\frac{\partial T}{\partial x}\frac{\partial N_i}{\partial x}\\
\frac{\partial}{\partial y}\left(k_y\frac{\partial T}{\partial y}\right)N_i&=\frac{\partial}{\partial y}\left(k_y\frac{\partial T}{\partial y}N_i\right)-k_y\frac{\partial T}{\partial y}\frac{\partial N_i}{\partial y}\\
\frac{\partial}{\partial z}\left(k_z\frac{\partial T}{\partial z}\right)N_i&=\frac{\partial}{\partial z}\left(k_z\frac{\partial T}{\partial z}N_i\right)-k_z\frac{\partial T}{\partial z}\frac{\partial N_i}{\partial z}
\end{aligned} \tag{7.73}$$

and the residual equations become

$$\iiint_V\left[\frac{\partial}{\partial x}\left(k_x\frac{\partial T}{\partial x}N_i\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial T}{\partial y}N_i\right)+\frac{\partial}{\partial z}\left(k_z\frac{\partial T}{\partial z}N_i\right)\right]\mathrm{d}V+\iiint_V QN_i\,\mathrm{d}V$$

$$=\iiint_V\left(k_x\frac{\partial T}{\partial x}\frac{\partial N_i}{\partial x}+k_y\frac{\partial T}{\partial y}\frac{\partial N_i}{\partial y}+k_z\frac{\partial T}{\partial z}\frac{\partial N_i}{\partial z}\right)\mathrm{d}V \qquad i=1,\ldots,M \tag{7.74}$$

The integral on the left side of Equation 7.74 contains a perfect differential in three dimensions and can be replaced by an integral over the surface of the volume using Green's theorem in three dimensions: *If* $F(x, y, z)$, $G(x, y, z)$, *and*

H(*x*, *y*, *z*) *are functions defined in a region of xyz space* (the element volume in our context), *then*

$$\iiint_V \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z}\right) \mathrm{d}V = \oiint_A (Fn_x + Gn_y + Hn_z)\, \mathrm{d}A \qquad (7.75)$$

where A is the surface area of the volume and n_x, n_y, n_z are the Cartesian components of the outward unit normal vector of the surface area. This theorem is the three-dimensional counterpart of integration by parts discussed earlier in this chapter.

Invoking Fourier's law and comparing Equation 7.75 to the first term of Equation 7.74, we have

$$-\oiint_A (q_x n_x + q_y n_y + q_z n_z) N_i \,\mathrm{d}A + \iiint_V Q N_i \,\mathrm{d}V$$
$$= \iiint_V \left(k_x \frac{\partial T}{\partial x}\frac{\partial N_i}{\partial x} + k_y \frac{\partial T}{\partial y}\frac{\partial N_i}{\partial y} + k_z \frac{\partial T}{\partial z}\frac{\partial N_i}{\partial z}\right) \mathrm{d}V \quad i = 1, \ldots, M \qquad (7.76)$$

Inserting the matrix form of Equation 7.71 and rearranging, we have

$$\iiint_V \left(k_x \frac{\partial [N]}{\partial x}\frac{\partial N_i}{\partial x} + k_y \frac{\partial [N]}{\partial y}\frac{\partial N_i}{\partial y} + k_z \frac{\partial [N]}{\partial z}\frac{\partial N_i}{\partial z}\right) \{T\}\,\mathrm{d}V$$
$$= \iiint_V Q N_i \,\mathrm{d}V - \oiint_A (q_x n_x + q_y n_y + q_z n_z) N_i \,\mathrm{d}A \qquad i = 1, \ldots, M \qquad (7.77)$$

Equation 7.77 represents a system of M algebraic equations in the M unknown nodal temperatures $\{T\}$. With the exception that convection effects are not included here, Equation 7.77 is analogous to the two-dimensional case represented by Equation 7.34. In matrix notation, the system of equations for the three-dimensional element formulation is

$$\iiint_V \left(k_x \frac{\partial [N]^T}{\partial x}\frac{\partial [N]}{\partial x} + k_y \frac{\partial [N]^T}{\partial y}\frac{\partial [N]}{\partial y} + k_z \frac{\partial [N]^T}{\partial z}\frac{\partial [N]}{\partial z}\right) \mathrm{d}V\{T\}$$
$$= \iiint_V Q[N]^T \mathrm{d}V - \oiint_A (q_x n_x + q_y n_y + q_z n_z)[N]^T \mathrm{d}A \qquad (7.78)$$

and Equation 7.76 is in the desired form

$$\left[k^{(e)}\right]\left\{T^{(e)}\right\} = \left\{f_Q^{(e)}\right\} + \left\{f_q^{(e)}\right\} \qquad (7.79)$$

Comparing the last two equations, the element conductance (stiffness) matrix is

$$\left[k^{(e)}\right] = \iiint_V \left(k_x \frac{\partial [N]^T}{\partial x}\frac{\partial [N]}{\partial x} + k_y \frac{\partial [N]^T}{\partial y}\frac{\partial [N]}{\partial y} + k_z \frac{\partial [N]^T}{\partial z}\frac{\partial [N]}{\partial z}\right) \mathrm{d}V \qquad (7.80)$$

the element force vector representing internal heat generation is

$$\left\{f_Q^{(e)}\right\} = \iiint\limits_V Q[N]^T \, \mathrm{d}V \tag{7.81}$$

and the element nodal force vector associated with heat flux across the element surface area is

$$\left\{f_q^{(e)}\right\} = -\oiint\limits_A (q_x n_x + q_y n_y + q_z n_z)[N]^T \, \mathrm{d}A \tag{7.82}$$

7.6.1 System Assembly and Boundary Conditions

The procedure for assembling the global equations for a three-dimensional model for heat transfer analysis is identical to that of one- and two-dimensional problems. The element type is selected (tetrahedral, brick, quadrilateral solid, for example) based on geometric considerations, primarily. The volume is then divided into a mesh of elements by first defining nodes (in the global coordinate system) throughout the volume then each element by the sequence and number of nodes required for the element type. Element-to-global nodal correspondence relations are then determined for each element, and the global stiffness (conductance) matrix is assembled. Similarly, the global force vector is assembled by adding element contributions at nodes common to two or more elements. The latter procedure is straightforward in the case of internal generation, as given by Equation 7.81. However, in the case of the element gradient terms, Equation 7.82, the procedure is best described in terms of the global boundary conditions.

In the case of three-dimensional heat transfer, we have the same three types of boundary conditions as in two dimensions: (1) specified temperatures, (2) specified heat flux, and (3) convection conditions. The first case, specified temperatures, is taken into account in the usual manner, by reducing the system equations by simply substituting the known nodal temperatures into the system equations. The latter two cases involve only elements that have surfaces (element faces) on the outside surface of the global volume. To illustrate, Figure 7.17a shows two brick elements that share a common face in an assembled finite element model. For convenience, we take the common face to be perpendicular to the x axis. In Figure 7.17b, the two elements are shown separately with the associated normal vector components identified for the shared faces. For steady-state heat transfer, the heat flux across the face is the same for each element and, since the unit normal vectors are opposite, the gradient force terms cancel. The result is completely analogous to internal forces in a structural problem via Newton's third law of action and reaction. Therefore, on interelement boundaries (which are areas for three-dimensional elements), the element force terms defined by Equation 7.82 sum to zero in the global assembly process.

What of the element surface areas that are part of the surface area of the volume being modeled? Generally, these outside areas are subjected to convection

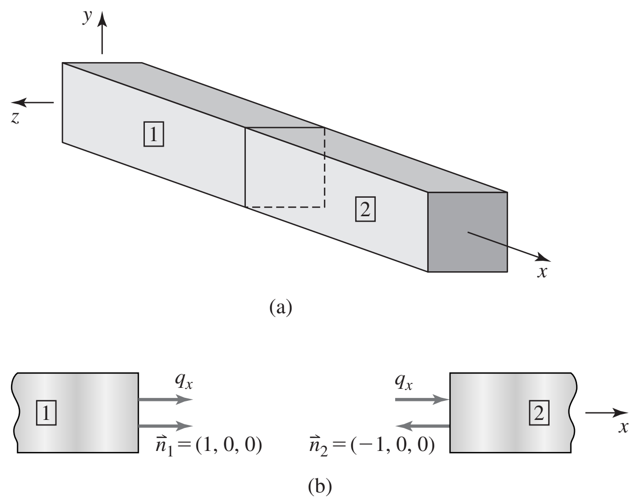

Figure 7.17
(a) Common face in two 3-D elements. (b) Edge view of common face, illustrating cancellation of conduction gradient terms.

conditions. For such convection boundary conditions, the flux conditions of Equation 7.82 must be in balance with the convection from the area of concern. Mathematically, the condition is expressed as

$$\left\{f_q^{(e)}\right\} = -\oiint_A (q_x n_x + q_y n_y + q_z n_z)[N]^T \,\mathrm{d}A = -\oiint_A q_n n[N]^T \,\mathrm{d}A$$
$$= -\oiint_A h\left(T^{(e)} - T_a\right)[N]^T \,\mathrm{d}A \tag{7.83}$$

where q_n is the flux normal to the surface area A of a specific element face on the global boundary and n is the unit outward normal vector to that face. As in two-dimensional analysis, the convection term in the rightmost integral of Equation 7.83 adds to the stiffness matrix when the expression for $T^{(e)}$ in terms of interpolation functions and nodal temperatures is substituted. Similarly, the ambient temperature terms add to the forcing function vector.

In most commercial finite element software packages, the three-dimensional heat transfer elements available do *not* explicitly consider the gradient force vector represented by Equation 7.82. Instead, such programs compute the system (global) stiffness matrix on the basis of conductance only and rely on the user to specify the flux or convection boundary conditions (and the specified temperature conditions, of course) as part of the loading (input) data.

Owing to the algebraic volume of calculation required, examples of general three-dimensional heat transfer are not presented here. A few three-dimensional problems are included in the end-of-chapter exercises and are intended to be solved by digital computer techniques.

7.7 AXISYMMETRIC HEAT TRANSFER

Chapter 6 illustrated the approach for utilizing two-dimensional elements and associated interpolation functions for axisymmetric problems. Here, we illustrate the formulation of finite elements to solve problems in axisymmetric heat transfer. Illustrated in Figure 7.18 is a body of revolution subjected to heat input at its base, and the heat input is assumed to be symmetric about the axis of revolution. Think of the situation as a cylindrical vessel heated by a source, such as a gas flame. This situation could, for example, represent a small crucible for melting metal prior to casting.

As an axisymmetric problem is three-dimensional, the basic governing equation is Equation 7.70, restated here under the assumption of homogeneity, so that $k_x = k_y = k_z = k$, as

$$k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial T^2}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + Q = 0 \tag{7.84}$$

Equation 7.84 is applicable to steady-state conduction only and is expressed in rectangular coordinates. For axisymmetric problems, use of a cylindrical coordinate system (r, θ, z) is much more amenable to formulating the problem. To convert to cylindrical coordinates, the partial derivatives with respect to x and y in Equation 7.84 must be converted mathematically into the corresponding partial derivatives with respect to radial coordinate r and tangential (circumferential)

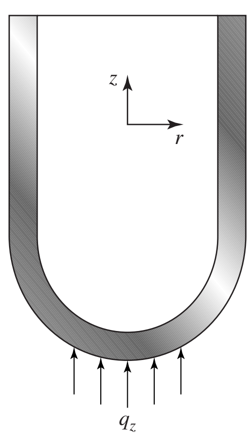

Figure 7.18 An axisymmetric heat transfer problem. All properties and inputs are symmetric about the z axis.

coordinate θ. In the following development, we present the general approach but leave the details as an end-of-chapter exercise.

The basic relations between the rectangular coordinates x, y and the cylindrical (polar) coordinates r, θ are

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \tag{7.85}$$

and inversely,

$$\begin{aligned} r^2 &= x^2 + y^2 \\ \tan\theta &= \frac{y}{x} \end{aligned} \tag{7.86}$$

Per the chain rule of differentiation, we have

$$\begin{aligned} \frac{\partial T}{\partial x} &= \frac{\partial T}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial T}{\partial \theta}\frac{\partial \theta}{\partial x} \\ \frac{\partial T}{\partial y} &= \frac{\partial T}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial T}{\partial \theta}\frac{\partial \theta}{\partial y} \end{aligned} \tag{7.87}$$

By implicit differentiation of Equation 7.86,

$$\begin{aligned} 2r\frac{\partial r}{\partial x} &= 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta \\ 2r\frac{\partial r}{\partial y} &= 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta \\ \frac{1}{\sec^2\theta}\frac{\partial \theta}{\partial x} &= -\frac{y}{x^2} \Rightarrow \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r} \\ \frac{1}{\sec^2\theta}\frac{\partial \theta}{\partial y} &= \frac{1}{x} \Rightarrow \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r} \end{aligned} \tag{7.88}$$

so that Equation 7.87 becomes

$$\begin{aligned} \frac{\partial T}{\partial x} &= \cos\theta\frac{\partial T}{\partial r} - \frac{\sin\theta}{r}\frac{\partial T}{\partial \theta} \\ \frac{\partial T}{\partial y} &= \sin\theta\frac{\partial T}{\partial r} + \frac{\cos\theta}{r}\frac{\partial T}{\partial \theta} \end{aligned} \tag{7.89}$$

For the second partial derivatives, we have

$$\begin{aligned} \frac{\partial^2 T}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) = \cos\theta\frac{\partial}{\partial r}\left(\frac{\partial T}{\partial x}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial T}{\partial x}\right) \\ \frac{\partial^2 T}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}\right) = \sin\theta\frac{\partial}{\partial r}\left(\frac{\partial T}{\partial y}\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial T}{\partial y}\right) \end{aligned} \tag{7.90}$$

Applying Equation 7.89 to the operations indicated in Equation 7.90 yields (with appropriate use of trigonometric identities)

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} \tag{7.91}$$

where the derivation represents a general change of coordinates. To relate to an axisymmetric problem, recall that there is no dependence on the tangential coordinate θ. Consequently, when Equations 7.84 and 7.91 are combined, the governing equation for axisymmetric heat transfer is

$$k\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}\right)+Q=0 \tag{7.92}$$

and, of course, note the absence of the tangential coordinate.

7.7.1 Finite Element Formulation

Per the general procedure, the total volume of the axisymmetric domain is discretized into finite elements. In each element, the temperature distribution is expressed in terms of the nodal temperatures and interpolation functions as

$$T^{(e)}=\sum_{i=1}^{M} N_i(r,z)T_i^{(e)} \tag{7.93}$$

where, as usual, M is the number of element nodes. Note particularly that the interpolation functions vary with radial coordinate r and axial coordinate z. Application of Galerkin's method using Equations 7.92 and 7.93 yields the residual equations

$$\iiint_V \left[k\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}\right)+Q\right]N_i r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z=0 \qquad i=1,\ldots,M \tag{7.94}$$

Observing that, for the axisymmetric case, the integrand is independent of the tangential coordinate θ, Equation 7.94 becomes

$$2\pi\iint_{A^{(e)}} \left[k\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}\right)+Q\right]N_i r\,\mathrm{d}r\,\mathrm{d}z=0 \qquad i=1,\ldots,M \tag{7.95}$$

where $A^{(e)}$ is the area of the element in the rz plane. The first two terms of the integrand can be combined to obtain

$$2\pi\iint_{A^{(e)}} \left[k\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\partial^2 T}{\partial z^2}\right)+Q\right]N_i r\,\mathrm{d}r\,\mathrm{d}z=0 \qquad i=1,\ldots,M \tag{7.96}$$

Observing that r is independent of z, Equation 7.96 becomes

$$2\pi \iint_{A^{(e)}} \left[k \left[\frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r} \right) + \frac{\partial}{\partial z}\left(r \left(\frac{\partial T}{\partial z} \right) \right) \right] + Qr \right] N_i \, \mathrm{d}r \, \mathrm{d}z = 0$$

$$i = 1, \ldots, M \qquad (7.97)$$

As in previous developments, we invoke the chain rule of differentiation as, for example,

$$\frac{\partial}{\partial r}\left(r N_i \frac{\partial T}{\partial r} \right) = N_i \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r} \right) + r \frac{\partial T}{\partial r}\frac{\partial N_i}{\partial r} \Rightarrow N_i \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r} \right)$$

$$= \frac{\partial}{\partial r}\left(r N_i \frac{\partial T}{\partial r} \right) - r \frac{\partial T}{\partial r}\frac{\partial N_i}{\partial r} \qquad i = 1, \ldots, M \qquad (7.98)$$

Noting that Equation 7.98 is also applicable to the z variable, the residual equations represented by Equation 7.97 can be written as

$$2\pi \iint_{A^{(e)}} k \left[\frac{\partial}{\partial r}\left(r N_i \frac{\partial T}{\partial r} \right) + \frac{\partial}{\partial z}\left(r N_i \frac{\partial T}{\partial z} \right) \right] \mathrm{d}r \, \mathrm{d}z + 2\pi \iint_{A^{(E)}} Q N_i r \, \mathrm{d}r \, \mathrm{d}z$$

$$= 2\pi \iint_{A^{(e)}} k \left(\frac{\partial T}{\partial r}\frac{\partial N_i}{\partial r} + \frac{\partial T}{\partial z}\frac{\partial N_i}{\partial z} \right) r \, \mathrm{d}r \, \mathrm{d}z \qquad i = 1, \ldots, M \qquad (7.99)$$

The first integrand on the left side of Equation 7.99 is a perfect differential in two dimensions, and the Green-Gauss theorem can be applied to obtain

$$2\pi \oint_{S^{(e)}} \left(k \frac{\partial T}{\partial r} n_r + k \frac{\partial T}{\partial z} n_z \right) r N_i \, \mathrm{d}S + 2\pi \iint_{A^{(e)}} Q N_i r \, \mathrm{d}r \, \mathrm{d}z$$

$$= 2\pi \iint_{A^{(e)}} k \left(\frac{\partial T}{\partial r}\frac{\partial N_i}{\partial r} + \frac{\partial T}{\partial z}\frac{\partial N_i}{\partial z} \right) r \, \mathrm{d}r \, \mathrm{d}z \qquad i = 1, \ldots, M \qquad (7.100)$$

where S is the boundary (periphery) of the element and n_r and n_z are the radial and axial components of the outward unit vector normal to the boundary. Applying Fourier's law in cylindrical coordinates,

$$\begin{aligned} q_r &= -k \frac{\partial T}{\partial r} \\ q_z &= -k \frac{\partial T}{\partial z} \end{aligned} \qquad (7.101)$$

and noting the analogy with Equation 7.33, we rewrite Equation 7.100 as

$$2\pi k \iint_{A^{(e)}} \left(\frac{\partial T}{\partial r}\frac{\partial N_i}{\partial r} + \frac{\partial T}{\partial z}\frac{\partial N_i}{\partial z} \right) r \, \mathrm{d}r \, \mathrm{d}z$$

$$= 2\pi \iint_{A^{(e)}} Q N_i r \, \mathrm{d}r \, \mathrm{d}z - 2\pi \oint_{S^{(e)}} q_s n_s N_i r \, \mathrm{d}S \qquad i = 1, \ldots, M \qquad (7.102)$$

The common term 2π could be omitted, but we leave it as a reminder of the three-dimensional nature of an axisymmetric problem. In particular, note that this term, in conjunction with r in the integrand of the last integral on the right-hand side of Equation 7.102, reinforces the fact that element boundaries are actually surfaces of revolution.

Noting that Equation 7.102 represents a system of M equations, the form of the system is that of

$$\left[k^{(e)}\right]\left\{T^{(e)}\right\} = \left\{f_Q^{(e)}\right\} + \left\{f_g^{(e)}\right\} \tag{7.103}$$

where $[k^{(e)}]$ is the element conductance matrix having individual terms defined by

$$k_{ij} = 2\pi k \iint_{A^{(e)}} \left(\frac{\partial N_i}{\partial r}\frac{\partial N_j}{\partial r} + \frac{\partial N_i}{\partial z}\frac{\partial N_j}{\partial z}\right) r \, \mathrm{d}r \, \mathrm{d}z \qquad i, j = 1, \ldots, M \tag{7.104}$$

and $\{T^{(e)}\}$ is the column matrix (vector) of element nodal temperatures per Equation 7.93. The element forcing functions include the internal heat generation terms given by

$$\left\{f_Q^{(e)}\right\} = 2\pi \iint_{A^{(e)}} Q\,\{N\}\, r \, \mathrm{d}r \, \mathrm{d}z \tag{7.105}$$

and the boundary gradient (flux) components

$$\left\{f_g^{(e)}\right\} = -2\pi \oint_{S^{(e)}} q_s n_s \{N\}\, r \, \mathrm{d}S \tag{7.106}$$

As has been discussed for other cases, on boundaries common to two elements, the flux terms are self-canceling in the model assembly procedure. Therefore, Equation 7.106 is applicable to element boundaries on a free surface. For such surfaces, the boundary conditions are of one of the three types delineated in previous sections: specified temperature, specified heat flux, or convection conditions.

EXAMPLE 7.10

Calculate the terms of the conductance matrix for an axisymmetric element based on the three-node plane triangular element.

■ Solution

The element and nodal coordinates are as shown in Figure 7.19. From the discussions in Chapter 6, if we are to derive the interpolation functions from basic principles, we first express the temperature variation throughout the element as

$$T(r, z) = a_0 + a_1 r + a_2 z = N_1(r, z)T_1 + N_2(r, z)T_2 + N_3(r, z)T_3$$

apply the nodal conditions, and solve for the constants. Rearranging the results in terms of nodal temperatures then reveals the interpolation functions. However, the results are exactly the same as those of Chapter 6, if we simply replace x and y with r and z, so that

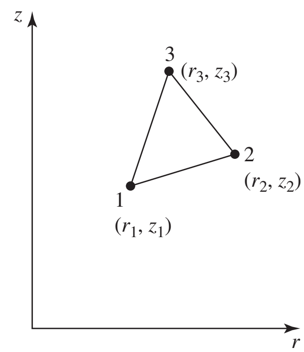

Figure 7.19 Cross section of a three-node axisymmetric element. Recall that the element is a body of revolution.

the interpolation functions are of the form

$$N_1(r, z) = \frac{1}{2A^{(e)}}(b_1 + c_1 r + d_1 z)$$
$$N_2(r, z) = \frac{1}{2A^{(e)}}(b_2 + c_2 r + d_2 z)$$
$$N_3(r, z) = \frac{1}{2A^{(e)}}(b_3 + c_3 r + d_3 z)$$

where

$$\begin{array}{lll} b_1 = r_2 z_3 - r_3 z_2 & b_2 = r_3 z_1 - r_1 z_3 & b_3 = r_1 z_2 - r_2 z_1 \\ c_1 = z_2 - z_3 & c_2 = z_3 - z_1 & c_3 = z_1 - z_2 \\ d_1 = r_3 - r_1 & d_2 = r_1 - r_3 & d_3 = r_2 - r_1 \end{array}$$

and $A^{(e)}$ is the area of the element in the rz plane.

Since the interpolation functions are linear, the partial derivatives are constants, so Equation 7.102 becomes

$$k_{ij} = 2\pi k \left(\frac{\partial N_i}{\partial r}\frac{\partial N_j}{\partial r} + \frac{\partial N_i}{\partial z}\frac{\partial N_j}{\partial z} \right) \iint_{A^{(e)}} r \, \mathrm{d}r \, \mathrm{d}z$$
$$= \frac{\pi k}{2\left(A^{(e)}\right)^2}(c_i c_j + d_i d_j) \iint_{A^{(e)}} r \, \mathrm{d}r \, \mathrm{d}z \qquad i, j = 1, 3$$

Recalling from elementary statics that the integral in the last equation represents the first moment of the area of the element about the z axis, we have

$$\iint_{A^{(e)}} r \, \mathrm{d}r \, \mathrm{d}z = \bar{r} A^{(e)}$$

where $\bar{r}$ is the radial coordinate of the element centroid. The components of the conductance matrix are then

$$k_{ij} = \frac{\pi k \bar{r}}{2A^{(e)}}(c_i c_j + d_i d_j) \qquad i, j = 1, 3$$

and the symmetry of the conductance matrix is evident.

7.8 TIME-DEPENDENT HEAT TRANSFER

The treatment of finite element analysis of heat transfer has, to this point, been limited to cases of steady-state conditions. No time dependence is included in such analyses, as we have assumed conditions such that a steady state is reached, and the transient conditions are not of interest. Certainly, transient, time-dependent effects are often quite important, and such effects determine whether a steady state is achieved and what that steady state will be. To illustrate time-dependent heat transfer in the context of finite element analysis, the one-dimensional case is discussed here.

The case of one-dimensional conduction without convection is detailed in Chapter 5. The governing equation, by consideration of energy balance in a control volume, Equation 5.54, is

$$q_x A\,\mathrm{d}t + QA\,\mathrm{d}x\,\mathrm{d}t = \Delta U + \left(q_x + \frac{\partial T}{\partial x}\,\mathrm{d}x\right) A\,\mathrm{d}t \tag{7.107}$$

where the temperature distribution $T(x, t)$ is now assumed to be dependent on both position and time. Further, the change in internal energy ΔU is not zero. Rather, the increase in internal energy during a small time interval is described by Equation 5.56, and the differential equation governing the temperature distribution, Equation 5.58, is

$$k_x \frac{\partial^2 T}{\partial x^2} + Q = c\rho \frac{\partial T}{\partial t} \tag{7.108}$$

where c and ρ denote material specific heat and density, respectively, and t is time. For time-dependent conduction, the governing equation is a second-order partial differential equation with constant coefficients.

Application of the finite element method for solution of Equation 7.108 proceeds by dividing the problem domain into finite-length, one-dimensional elements and discretizing the temperature distribution within each element as

$$T(x, t) = N_1(x)T_1(t) + N_2(x)T_2(t) = [N(x)]\{T(t)\} \tag{7.109}$$

which is the same as Equation 5.60 with the notable exception that *the nodal temperatures are functions of time.*

Let us now apply Galerkin's finite element method to Equation 7.108 to obtain the residual equations

$$\int_{x_1}^{x_2} \left(k_x \frac{\partial^2 T}{\partial x^2} + Q - c\rho \frac{\partial T}{\partial t}\right) N_i(x) A\,\mathrm{d}x \qquad i = 1, 2 \tag{7.110}$$

Noting that

$$N_i \frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(N_i \frac{\partial T}{\partial x}\right) - \frac{\partial N_i}{\partial x}\frac{\partial T}{\partial x} \tag{7.111}$$

the residual equations can be rearranged and expressed as

$$\int_{x_1}^{x_2} k_x \frac{\partial N_i}{\partial x}\frac{\partial T}{\partial x} A \,\mathrm{d}z + \int_{x_1}^{x_2} c\rho \frac{\partial T}{\partial t} N_i A \,\mathrm{d}x$$
$$= \int_{x_1}^{x_2} Q N_i A \,\mathrm{d}x + \int_{x_1}^{x_2} k_x \frac{\partial}{\partial x}\left(N_i \frac{\partial T}{\partial x}\right) A \,\mathrm{d}x \qquad i = 1, 2 \qquad (7.112)$$

Comparing Equation 7.112 to Equations 5.63 and 5.64, we observe that the first integral on the left includes the conductance matrix, the first integral on the right is the forcing function associated with internal heat generation, and the second integral on the right represents the gradient boundary conditions. Utilizing Equation 7.109, Equation 7.112 can be written in detailed matrix form as

$$c\,\rho A \int_{x_1}^{x_2} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} [N_1 \quad N_2]\,\mathrm{d}x \begin{Bmatrix} \dot{T}_1 \\ \dot{T}_2 \end{Bmatrix}$$
$$+ k_x A \int_{x_1}^{x_2} \begin{bmatrix} \dfrac{\mathrm{d}N_1}{\mathrm{d}x} \\ \dfrac{\mathrm{d}N_2}{\mathrm{d}x} \end{bmatrix} \begin{bmatrix} \dfrac{\mathrm{d}N_1}{\mathrm{d}x} & \dfrac{\mathrm{d}N_2}{\mathrm{d}x} \end{bmatrix} \mathrm{d}x \begin{Bmatrix} T_1(t) \\ T_2(t) \end{Bmatrix} = \{f_Q\} + \{f_g\} \qquad (7.113)$$

where the dot denotes differentiation with respect to time. Note that the derivatives of the interpolation functions have now been expressed as ordinary derivatives, as appropriate. Equation 7.113 is most often expressed as

$$\left[C^{(e)}\right]\left\{\dot{T}^{(e)}\right\} + \left[k^{(e)}\right]\left\{T^{(e)}\right\} = \left\{f_Q^{(e)}\right\} + \left\{f_g^{(e)}\right\} \qquad (7.114)$$

where $[C^{(e)}]$ is the element *capacitance matrix* defined by

$$\left[C^{(e)}\right] = c\,\rho A \int_{x_1}^{x_2} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} [N_1 \quad N_2]\,\mathrm{d}x = c\,\rho A \int_{x_1}^{x_2} [N]^T [N]\,\mathrm{d}x \qquad (7.115)$$

and, as implied by the name, indicates the capacity of the element for heat storage. The capacitance matrix defined by Equation 7.115 is known as the *consistent* capacitance matrix. The consistent capacitance matrix is so called because it is formulated on the basis of the same interpolation functions used to describe the spatial distribution of temperature. In our approach, using Galerkin's method, the consistent matrix is a natural result of the mathematical procedure. An alternate approach produces a so-called lumped capacitance matrix. Whereas the consistent matrix distributes the capacitance throughout the element by virtue of the interpolation functions, the lumped capacitance matrix ascribes the storage capacity strictly to the nodes independently. The difference in the two approaches is discussed in terms of heat transfer and, in more detail, in Chapter 10 in the context of structural dynamics.

The model assembly procedure for a transient heat transfer problem is exactly the same as for a steady-state problem, with the notable exception that we must also assemble a global capacitance matrix. The rules are the same. Element nodes are assigned to global nodes and the element capacitance matrix terms are added to the appropriate global positions in the global capacitance matrix, as with the conductance matrix terms. Hence, on system assembly, we obtain the global equations

$$[C]\{\dot{T}\} + [K]\{T\} = \{F_Q\} + \{F_g\} \tag{7.116}$$

where we must recall that the gradient force vector $\{F_g\}$ is composed of either (1) unknown heat flux values to be determined (unknown reactions) or (2) convection terms to be equilibrated with the flux at a boundary node.

7.8.1 Finite Difference Methods for the Transient Response: Initial Conditions

The finite element discretization procedure has reduced the one-dimensional transient heat transfer problem to algebraic terms in the spatial variable via the interpolation functions. Yet Equation 7.116 represents a set of ordinary, coupled, first-order differential equations in time. Consequently, as opposed to the steady-state case, there is not *a* solution but multiple solutions as the system responds to time-dependent conditions. The boundary conditions for a transient problem are of the three types discussed for the steady-state case: specified nodal temperatures, specified heat flux, or convection conditions. However, note that the boundary conditions may also be time dependent. For example, a specified nodal temperature could increase linearly with time to some specified final value. In addition, an internal heat generation source Q may also vary with time.

A commonly used approach to obtaining solutions for ordinary differential equations of the form of Equation 7.116 is the *finite difference method*. As discussed briefly in Chapter 1, the finite difference method is based on approximating derivatives of a function as incremental changes in the value of the function corresponding to finite changes in the value of the independent variable. Recall that the first derivative of a function $f(t)$ is defined by

$$\dot{f} = \frac{\mathrm{d}f}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} \tag{7.117}$$

Instead of requiring Δt to approach zero, we obtain an approximation to the value of the derivative by using a small, nonzero value of Δt to obtain

$$\dot{f} \cong \frac{f(t + \Delta t) - f(t)}{\Delta t} \tag{7.118}$$

and the selected value of Δt is known as the *time step*.

To apply the procedure to transient heat transfer, we approximate the time derivative of the nodal temperature matrix as

$$\{\dot{T}\} \cong \frac{\{T(t + \Delta t)\} - \{T(t)\}}{\Delta t} \tag{7.119}$$

Substituting, Equation 7.116 becomes

$$[C]\frac{\{T(t+\Delta t)\}-\{T(t)\}}{\Delta t}+[K]\{T(t)\}=\{F_Q(t)\}+\{F_g(t)\} \tag{7.120}$$

Note that, if the nodal temperatures are known at time t and the forcing functions are evaluated at time t, Equation 7.120 can be solved, algebraically, for the nodal temperatures at time $t+\Delta t$. Denoting the time at the ith time step as $t_i = i(\Delta t)$, $i = 0, 1, 2, \ldots,$ we obtain

$$[C]\{T(t_{i+1})\}=[C]\{T(t_i)\}-[K]\{T(t_i)\}\,\Delta t+\{F_Q(t_i)\}\Delta t+\{F_g(t_i)\}\Delta t \tag{7.121}$$

as the system of algebraic equations that can be solved for $\{T(t_{i+1})\}$. Formally, the solution is obtained by multiplying Equation 7.121 by the inverse of the capacitance matrix. For large matrices common to finite element models, inverting the matrix is very inefficient, so other techniques such as Gaussian elimination are more often used. Note, however, that the system of algebraic equations given by Equation 7.121 must be solved only once to obtain an *explicit* solution for the nodal temperatures at time t_{i+1}.

The method just described is known as a *forward difference scheme* (also known as *Euler's method*) and Equation 7.121 is a *two-point recurrence relation*. If the state of the system (nodal temperatures and forcing functions) is known at one point in time, Equation 7.121 gives the state at the next point in time. Solving the system sequentially at increasing values of the independent variable is often referred to as *marching* in time. To begin the solution procedure, the state of the system must be known at $t = 0$. Therefore, the *initial conditions* must be specified in addition to the applicable boundary conditions. Recall that the general solution to an ordinary, first-order differential equation contains one constant of integration. As we have one such equation corresponding to each nodal temperature, the value of each nodal temperature must be specified at time zero. If the initial conditions are so known, the recurrence relation can be used to compute succeeding nodal temperatures. Prior to discussing other schemes and the ramifications of time step selection, the following simple example is presented.

EXAMPLE 7.11

Figure 7.20a shows a cylindrical rod having diameter of 12 mm and length of 100 mm. The pin is of a material having thermal conductivity 230 W/(m-°C), specific heat 900 J/(kg-°C), and density 2700 kg/m^3. The right-hand end of the rod is held in contact with a medium at a constant temperature of 30°C. At time zero, the entire rod is at a temperature of 30°C when a heat source is applied to the left end, bringing the temperature of the left end immediately to 80°C and maintaining that temperature indefinitely. Using the forward difference method and four two-node elements, determine both transient and steady-state temperature distributions in the rod. No internal heat is generated.

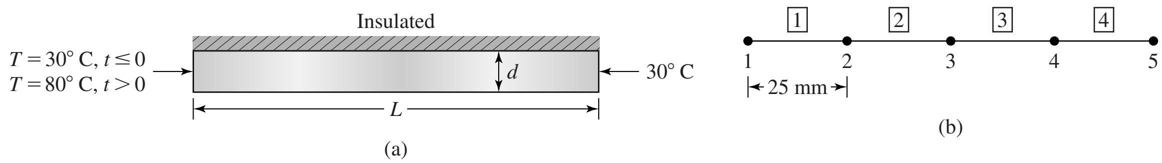

Figure 7.20
(a) Cylindrical rod of Example 7.11. (b) Node and element numbers.

■ Solution

In the solution for this example, we set up the general procedure then present the results for one solution using one time step for the transient portion. The node numbers and element numbers are as shown in Figure 7.20b. Since the length and area of each element are the same, we compute the element capacitance matrix as

$$[C^{(e)}] = \frac{c\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{900(2700)\frac{\pi}{4}(0.012)^2(0.025)}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2.2902 & 1.1451 \\ 1.1451 & 2.2902 \end{bmatrix} \text{J/}^\circ\text{C}$$

where we have implicitly performed the integrations indicated in Equation 7.115 and leave the details as an end-of-chapter exercise. Similarly, the element conductance matrix is

$$[k^{(e)}] = \frac{kA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{200\frac{\pi}{4}(0.012)^2}{0.025}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.9408 & -0.9408 \\ -0.9408 & 0.9408 \end{bmatrix} \text{W/}^\circ\text{C}$$

For the one-dimensional case with uniform geometry and material properties, the system assembly is straightforward and results in the global matrices

$$[C] = \begin{bmatrix} 2.2902 & 1.1451 & 0 & 0 & 0 \\ 1.1451 & 4.5804 & 1.1451 & 0 & 0 \\ 0 & 1.1451 & 4.5804 & 1.1451 & 0 \\ 0 & 0 & 1.1451 & 4.5804 & 1.1451 \\ 0 & 0 & 0 & 1.1451 & 2.2902 \end{bmatrix}$$

$$[K] = \begin{bmatrix} 0.9408 & -0.9408 & 0 & 0 & 0 \\ -0.9408 & 1.8816 & -0.9408 & 0 & 0 \\ 0 & -0.9408 & 1.8816 & -0.9408 & 0 \\ 0 & 0 & -0.9408 & 1.8816 & -0.9408 \\ 0 & 0 & 0 & -0.9408 & 0.9408 \end{bmatrix}$$

As no internal heat is generated $\{F_Q\} = 0$ and, as we have specified boundary temperatures, the flux forcing term is an unknown. Note that, in the transient case, the flux terms at the boundaries (the "reactions") are time dependent and can be computed at each time step, as will be explained. Hence, the gradient "force vector" is

$$\{F_g\} = \begin{Bmatrix} q_1 A \\ 0 \\ 0 \\ 0 \\ -q_5 A \end{Bmatrix}$$

Having taken care of the boundary conditions, we now consider the initial conditions and examine the totality of the conditions on the solution procedure. It should be clear that, since we have the temperature of two nodes specified, the desired solution should provide the temperatures of the other three nodes and, therefore, should be a 3×3 system. The reduction to the 3×3 system is accomplished via the following observations:

1. If $T_1 = 80°\text{C} = \text{constant}$, then $\dot{T}_1 = 0$.
2. If $T_5 = 30°\text{C} = \text{constant}$, then $\dot{T}_5 = 0$.

The equations can be modified accordingly. In this example, the general equations become

$$[C]\begin{Bmatrix} 0 \\ \dot{T}_2 \\ \dot{T}_3 \\ \dot{T}_4 \\ 0 \end{Bmatrix} + [K]\begin{Bmatrix} 80 \\ T_2 \\ T_3 \\ T_4 \\ 30 \end{Bmatrix} = \begin{Bmatrix} q_1 A \\ 0 \\ 0 \\ 0 \\ -q_5 A \end{Bmatrix}$$

Consequently, the first and fifth equations become

$$1.1451\dot{T}_2 + 0.9408(80) - 0.9408T_2 = q_1 A$$

$$1.1451\dot{T}_4 - 0.9408T_4 + 0.9408(30) = -q_5 A$$

respectively. The three remaining equations are then written as

$$\begin{bmatrix} 4.5804 & 1.1451 & 0 \\ 1.1451 & 4.5804 & 1.1451 \\ 0 & 1.1451 & 4.5804 \end{bmatrix}\begin{Bmatrix} \dot{T}_2 \\ \dot{T}_3 \\ \dot{T}_4 \end{Bmatrix} + \begin{bmatrix} 1.8816 & -0.9408 & 0 \\ -0.9408 & 1.8816 & -0.9408 \\ 0 & -0.9408 & 1.8816 \end{bmatrix}\begin{Bmatrix} T_2 \\ T_3 \\ T_4 \end{Bmatrix}$$

$$= \begin{Bmatrix} 75.264 \\ 0 \\ 28.224 \end{Bmatrix}$$

For this example, the capacitance matrix is inverted (using a spreadsheet program) to obtain

$$[C]^{-1} = \begin{bmatrix} 0.2339 & -0.0624 & 0.0156 \\ -0.0624 & 0.2495 & -0.0624 \\ 0.0156 & -0.0624 & 0.2339 \end{bmatrix}$$

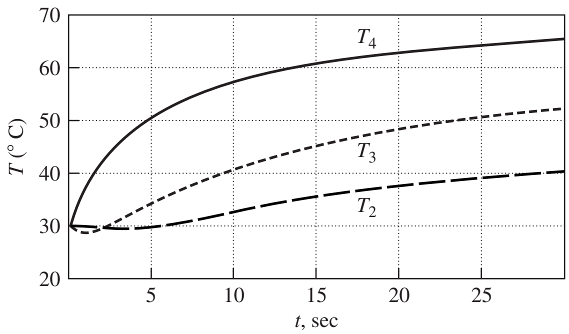

Figure 7.21 Time histories of the nodal temperatures.

where $[C]$ now represents the reduced 3 × 3 capacitance matrix. Utilizing Equation 7.121 and multiplying by $[C]^{-1}$ yields

$$\begin{Bmatrix} T_2 \\ T_3 \\ T_4 \end{Bmatrix}_{i+1} = \begin{Bmatrix} T_2 \\ T_3 \\ T_4 \end{Bmatrix}_i - \begin{bmatrix} 0.4988 & -0.3521 & 0.0880 \\ -0.3521 & 0.5869 & -0.3521 \\ 0.0880 & -0.3521 & 0.4988 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \\ T_4 \end{Bmatrix}_i \Delta t + \begin{Bmatrix} 18.0456 \\ -6.4558 \\ 7.7762 \end{Bmatrix} \Delta t$$

as the two-point recurrence relation.

Owing to the small matrix involved, the recurrence relation was programmed into a standard spreadsheet program using time step $\Delta t = 0.1$ sec. Calculations for nodal temperatures T_2, T_3, and T_4 are carried out until a steady state is reached. Time histories of each of the nodal temperature are shown in Figure 7.21. The figure shows that steady-state conditions $T_2 = 67.5°\text{C}$, $T_3 = 55°\text{C}$, and $T_4 = 42.5°\text{C}$ are attained in about 30 sec. Interestingly, the results also show that the temperatures of nodes 3 and 4 initially decrease. Such phenomena are physically unacceptable and associated with use of a consistent capacitance matrix, as is discussed in Chapter 10.

7.8.2 Central Difference and Backward Difference Methods

The forward difference method discussed previously and used in Example 7.11 is but one of three commonly used finite difference methods. The others are the backward difference method and the central difference method. Each of these is discussed in turn and a single two-point recurrence relation is developed incorporating the three methods.

In the *backward difference method,* the finite approximation to the first derivative at time t is expressed as

$$\dot{T}(t) \cong \frac{T(t) - T(t - \Delta t)}{\Delta t} \tag{7.122}$$

so that we, in effect, look back in time to approximate the derivative during the previous time step. Substituting this relation into Equation 7.116 gives

$$[C]\frac{\{T(t)\} - \{T(t - \Delta t)\}}{\Delta t} + [K]\{T(t)\} = \{F_Q(t)\} + \{F_g(t)\} \qquad (7.123)$$

In this method, we evaluate the nodal temperatures at time t based on the state of the system at time $t - \Delta t$, so we introduce the notation $t = t_i$, $t_{i-1} = t - \Delta t$, $i = 1, 2, 3, \ldots$. Using the described notation and rearranging, Equation 7.123 becomes

$$([C] + [K]\Delta t)\{T(t_i)\} = [C]\{T(t_{i-1})\} + F_Q(t_i)\Delta t + F_g(t_i)\Delta t \qquad i = 1, 2, 3, \ldots \qquad (7.124)$$

If the nodal temperatures are known at time t_{i-1}, Equation 7.124 can be solved for the nodal temperatures at the next time step (it is assumed that the forcing functions on the right-hand side are known and can be determined at t_i). Noting that the time index is relative, Equation 7.125 can also be expressed as

$$\begin{aligned}([C] + [K]\Delta t)\,&\{T(t_{i+1})\}\\ &= [C]\,\{T(t_i)\} + F_Q(t_i)\Delta t + F_g(t_i)\Delta t \qquad i = 0, 1, 2, \ldots \end{aligned} \qquad (7.125)$$

If we compare Equation 7.125 with Equation 7.121, we find that the major difference lies in the treatment of the conductance matrix. In the latter case, the effects of conductance are, in effect, updated during the time step. In the case of the forward difference method, Equation 7.121, the conductance effects are held constant at the previous time step. We also observe that Equation 7.125 cannot be solved at each time step by "simply" inverting the capacitance matrix. The coefficient matrix on the left-hand side changes at each time step; therefore, more efficient methods are generally used to solve Equation 7.125.

Another approach to approximation of the first derivative is the central difference method. As the name implies, the method is a compromise of sorts between forward and backward difference methods. In a central difference scheme, the dependent variable and all forcing functions are evaluated at the center (midpoint) of the time step. In other words, average values are used. In the context of transient heat transfer, the time derivative of temperature is still as approximated by Equation 7.119 but the other terms in Equation 7.120 are evaluated at the midpoint of the time step. Using this approach, Equation 7.120 becomes

$$\begin{aligned}[C]\frac{\{T(t + \Delta t)\} - \{T(t)\}}{\Delta t} &+ [K]\left\{\frac{T(t + \Delta t) + T(t)}{2}\right\}\\ &= \left\{\frac{F_Q(t + \Delta t) + F_Q(t)}{2}\right\} + \left\{\frac{F_g(t + \Delta t) + F_g(t)}{2}\right\}\end{aligned} \qquad (7.126)$$

The forcing functions on the right-hand side of Equation 7.126 are either known functions and can be evaluated or "reactions," which are subsequently computed via the constraint equations. The left-hand side of Equation 7.126 is now, however, quite different, in that the unknowns at each step $T_i(t + \Delta t)$ appear in both capacitance and conductance terms. Multiplying by Δt and rearranging Equation 7.126, we obtain

$$\left([C] + [K]\frac{\Delta t}{2}\right)\{T(t + \Delta t)\}$$
$$= \left([C] - [K]\frac{\Delta t}{2}\right)\{T(t)\} + \left\{\frac{F_Q(t + \Delta t) + F_Q(t)}{2}\right\} + \left\{\frac{F_g(t + \Delta t) + F_g(t)}{2}\right\} \tag{7.127}$$

Equation 7.127 can be solved for the unknown nodal temperatures at time $t + \Delta t$ and the "marching" solution can progress in time until a steady state is reached. The central difference methods is, in general, more accurate than the forward or backward difference method, in that it does not give preference to either temperatures at t or $t + \Delta t$ but, rather, gives equal credence to both.

In finite difference methods, the key parameter governing solution accuracy is the selected time step Δt. In a fashion similar to the finite element method, in which the smaller the elements are, physically, the better is the solution, the finite difference method converges more rapidly to the true solution as the time step is decreased. These ideas are amplified in Chapter 10, when we examine the dynamic behavior of structures.

7.9 CLOSING REMARKS

In Chapter 7, we expand the application of the finite element method into two- and three-dimensional, as well as axisymmetric, problems in heat transfer. While the majority of the chapter focuses on steady-state conditions, we also present the finite difference methods commonly used to examine transient effects. The basis of our approach is the Galerkin finite element method, and this text stays with that procedure, as it is so general in application. As we proceed into applications in fluid mechanics, solid mechanics, and structural dynamics in the following chapters, the Galerkin method is the basis for the development of many of the finite element models.

REFERENCES

1. Huebner, K. H., and E. A. Thornton. *The Finite Element Method for Engineers,* 2nd ed. New York: John Wiley and Sons, 1982.
2. Incropera, F. P., and D. P. DeWitt. *Introduction to Heat Transfer,* 3rd ed. New York: John Wiley and Sons, 1996.

PROBLEMS

7.1 For Example 7.1, determine the exact solution by integrating Equation 5.59 and applying the boundary conditions to evaluate the constants of integration.

7.2 Verify the convection-related terms in Equation 7.15 by direct integration.

7.3 For the data given in Example 7.4, use Gaussian quadrature with four integration points (two on r, two on s) to evaluate the terms of the stiffness matrix. Do your results agree with the values given in the example?

7.4 Using the computed nodal temperatures and heat flux values calculated in Example 7.5, perform a check calculation on the heat flow balance. That is, determine whether the heat input is in balance with the heat loss due to convection. How does this check indicate the accuracy of the finite element solution?

7.5 Consider the circular heat transfer pin shown in Figure P7.5. The base of the pin is held at constant temperature of 100°C (i.e., boiling water). The tip of the pin and its lateral surfaces undergo convection to a fluid at ambient temperature T_a. The convection coefficients for tip and lateral surfaces are equal. Given $k_x =$ 380 W/m-°C, $L = 8$ cm, $h = 2500$ W/m^2-°C, $d = 2$ cm, $T_a = 30$°C. Use a two-element finite element model with linear interpolation functions (i.e., a two-node element) to determine the nodal temperatures and the heat removal rate from the pin. Assume no internal heat generation.

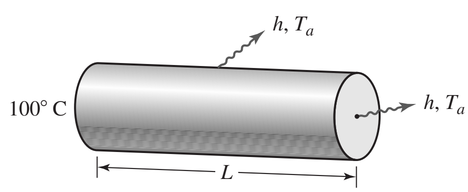

Figure P7.5

7.6 Repeat Problem 7.5 using four elements. Is convergence indicated?

7.7 The pin of Figure P7.5 represents a heating unit in a water heater. The base of the pin is held at fixed temperature 30°C. The pin is surrounded by flowing water at 55°C. Internal heat generation is to be taken as the constant value $Q = 25$ W/cm^3. All other data are as given in Problem 7.5. Use a two-element model to determine the nodal temperatures and the net heat flow rate from the pin.

7.8 Solve Problem 7.5 under the assumption that the pin has a square cross section 1 cm × 1 cm. How do the results compare in terms of heat removal rate?

7.9 The efficiency of the pin shown in Figure P7.5 can be defined in several ways. One way is to assume that the maximum heat transfer occurs when the entire pin is at the same temperature as the base (in Problem 7.5, 100°C), so that convection is maximized. We then write

$$q_{\max} = \int_0^L hP(T_b - T_a)\,\mathrm{d}x + hA(T_b - T_a)$$

where T_b represents the base temperature, P is the peripheral dimension, and A is cross-sectional area at the tip. The actual heat transfer is less than $q_{\max}$, so we define efficiency as

$$\eta = \frac{q_{act}}{q_{max}}$$

Use this definition to determine the efficiency of the pin of Problem 7.5.

7.10 Figure P7.10 represents one tube of an automotive engine's radiator. The engine coolant is circulated through the tube at a constant rate determined by the water pump. Cooling is primarily via convection from flowing air around the tube as a result of vehicle motion. Coolant enters the tube at a temperature of 195°F and the flow rate is 0.3 gallons per minute (specific weight is 68.5 lb/ft^3). The physical data are as follows: $L = 18$ in., $d = 0.3125$ in., $k_x = 225$ Btu/hr-ft-°F, $h = 37$ Btu/ft^2-hr-°F. Determine the stiffness matrix and load vector for an element of arbitrary length to use in modeling this problem, assuming steady-state conditions. Is the assumption of steady-state conditions reasonable?

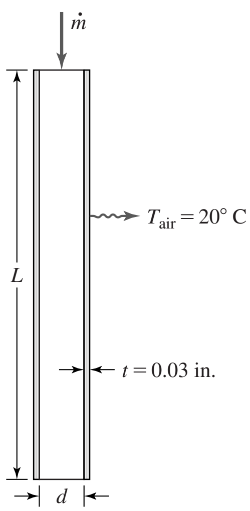

Figure P7.10

7.11 Use the results of Problem 7.10 to model the tube with four equal length elements and determine the nodal temperatures and the total heat flow from surface convection.

7.12 Consider the tapered heat transfer pin shown in Figure P7.12a. The base of the pin is held at constant temperature T_b, while the lateral surfaces and tip are surrounded by a fluid media held at constant temperature T_a. Conductance k_x and the convection coefficient h are known constants. Figure P7.12b shows the pin modeled as four tapered finite elements. Figure P7.12c shows the pin modeled as four constant cross-section elements with the area of each element equal to the average area of the actual pin. What are the pros and cons of the two modeling approaches? Keep in mind that use of four elements is only a starting point, many more elements are required to obtain a convergent solution. How does the previous statement affect your answer?

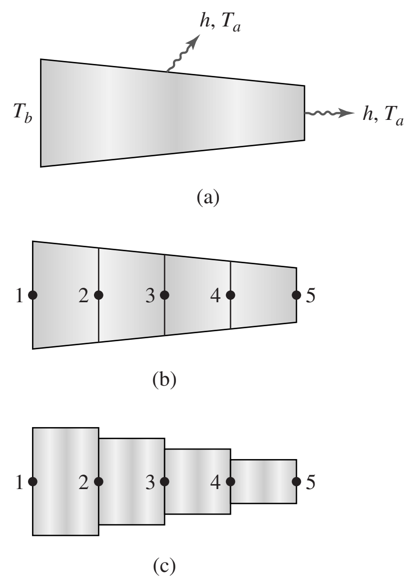

Figure P7.12

7.13 A cylindrical pin is constructed of a material for which the thermal conductivity decreases with temperature according to

$$k_x = k_0 - cT$$

and c is a positive constant. For this situation, show that the governing equation for steady-state, one-dimensional conduction with convection is

$$k_0 \frac{\mathrm{d}^2 T}{\mathrm{d}x^2} - \frac{\mathrm{d}}{\mathrm{d}x}\left(cT\frac{\mathrm{d}T}{\mathrm{d}x}\right) + Q = \frac{hP}{A}(T - T_a)$$

7.14 The governing equation for the situation described in Problem 7.13 is nonlinear. If the Galerkin finite element method is applied, an integral of the form

$$\int_{x_1}^{x_2} N_i \frac{\mathrm{d}}{\mathrm{d}x}\left(cT\frac{\mathrm{d}T}{\mathrm{d}x}\right)\mathrm{d}x$$

appears in the conductance matrix formulation. Integrate this term by parts and discuss the results in terms of boundary flux conditions and the conductance matrix.

7.15 A vertical wall of “sandwich” construction shown in Figure P7.15a is held at constant temperature $T_1 = 68°\text{F}$ on one surface and $T_2 = 28°\text{F}$ on the other surface. Using only three elements (one for each material), as in Figure P7.15b, determine the nodal temperatures and heat flux through the wall per unit area. The dimensions of the wall in the y and z directions are very large in comparison to wall thickness.

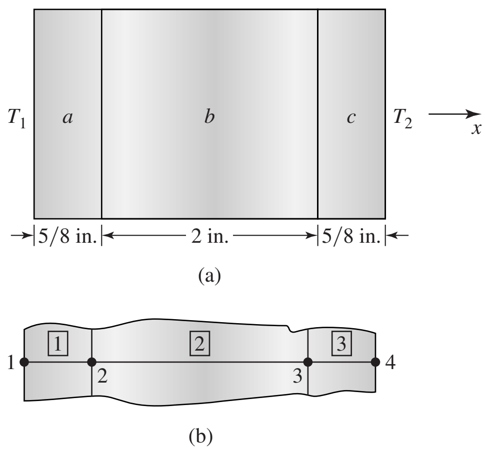

Figure P7.15

7.16 The wall of Problem 7.15 carries a centrally located electrical cable, which is to be treated as a line heat source of strength $Q^* = constant$, as shown in Figure P7.16. With this change, can the problem still be treated as one dimensional? If your answer is yes, solve the problem using three elements as in Problem 7.15. If your answer is no, explain.

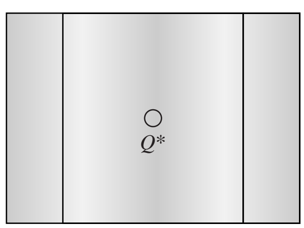

Figure P7.16

7.17 A common situation in polymer processing is depicted in Figure P7.17, which shows a "jacketed" pipe. The inner pipe is stainless steel having thermal conductivity k; the outer pipe is carbon steel and assumed to be perfectly insulated. The annular region between the pipes contains a heat transfer medium at constant temperature T_1. The inner pipe contains polymer material flowing

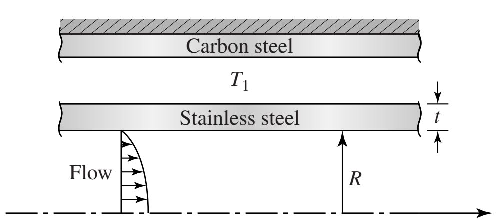

Figure P7.17

at constant mass flow rate $\dot{m}$. The convection coefficient between the stainless steel pipe wall and the polymer is h. Polymer specific heat c is also taken to be constant. Is this a one-dimensional problem? How would you solve this problem using the finite element method?

7.18 An office heater (often incorrectly called a *radiator,* since the heat transfer mode is convection) is composed of a central pipe containing heated water at constant temperature, as depicted in Figure P7.18. Several two-dimensional heat transfer fins are attached to the pipe as shown. The fins are equally spaced along the length of the pipe. Each fin has thickness of 0.125 in. and overall dimensions 4 in. × 4 in. Convection from the edges of the fins can be neglected. Consider the pipe as a point source $Q^* = 600$ Btu/hr-ft^2 and determine the net heat transfer to the ambient air at 20°C, if the convection coefficient is $h = 300$ Btu/(hr-ft^2-°F). Use four finite elements with linear interpolation functions (consider symmetry conditions here).

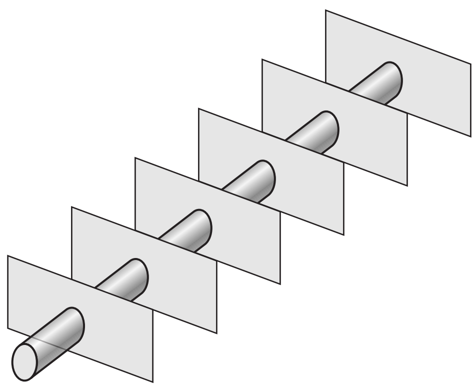

Figure P7.18

7.19 One who seriously considers the symmetry conditions of Problem 7.18 would realize that quarter symmetry exists and four elements represent 16 elements in the full problem domain. What are the boundary conditions for the symmetric model?

7.20 The rectangular fins shown in Figure P7.20 are mounted on a centrally located pipe carrying hot water. Temperature at the contact surface between fin and pipe is a constant T_1. For each case depicted, determine the applicable symmetry conditions and the boundary conditions applicable to a finite element model.

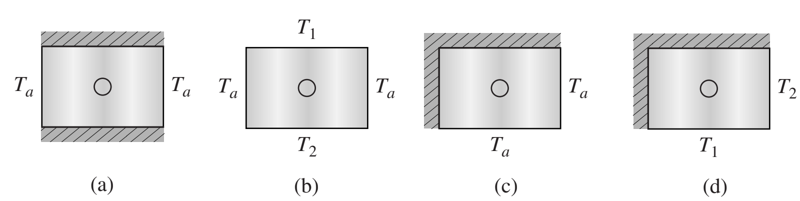

Figure P7.20

Note that the cross-hatched edges are perfectly insulated and it is assumed that the convection coefficients on all surfaces is the same constant.

7.21 Solve Example 7.5 using the two-element model in Figure 7.11b. How do the results compare to those of the four-element model?

7.22 The rectangular element shown in Figure P7.22 contains a line source of constant strength Q^* located at the element centroid. Determine $\{f_Q^{(e)}\}$.

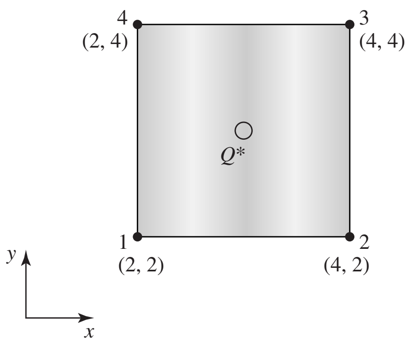

Figure P7.22

7.23 Determine the forcing function components of $\{f_Q^{(e)}\}$ for the axisymmetric element of Example 7.10 for the case of uniform internal heat generation Q.

7.24 A steel pipe (outside diameter of 60 mm and wall thickness of 5 mm) contains flowing water at constant temperature 80°C, as shown in Figure P7.24. The convection coefficient between the water and pipe is 2000 W/m^2-°C. The pipe is surrounded by air at 20°C, and the convection at the outer pipe surface is 20 W/m^2-°C. The thermal conductivity of the pipe material is 60 W/m-°C. Determine the stiffness matrices and nodal forcing functions for the two axisymmetric elements shown.

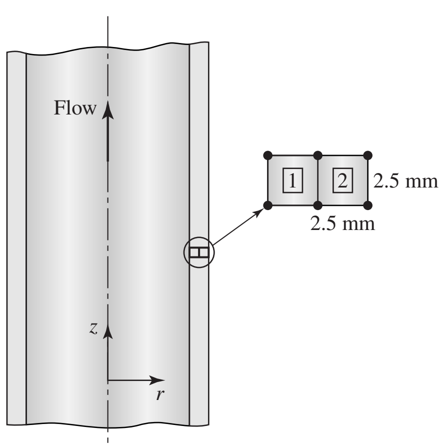

Figure P7.24

7.25 Assemble the global equations for the two elements of Problem 7.24. Use the global node numbers shown in Figure P7.25. Compute the nodal temperatures, and find the net heat flow (per unit surface area) into the surrounding air.

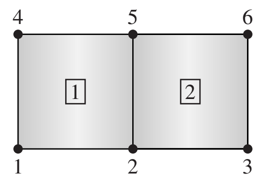

Figure P7.25

7.26 Solve the problem of Example 7.11 using a time step $\Delta t = 0.01$ sec. How do the results compare to those of the example solution?